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Solving Directly General Third Order Ordinary Differential Equations Using Two-Point Four Step Block Method

(Penyelesaian Terus Persamaan Pembezaan Biasa Am Peringkat Tiga Menggunakan Kaedah Blok Dua-Titik Empat Langkah)

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ABSTRACT

Two-point four step direct implicit block method is presented by applying the simple form of Adams- Moulton method for solving directly the general third order ordinary differential equations (ODEs) using variable step size. This method is implemented to get the solutions of initial value problems (IVPs) at two points simultaneously in a block using four backward steps. The numerical results showed that the performance of the developed method is better in terms of maximum error at all tested tolerances and lesser total number of steps as the tolerances getting smaller compared to the existence direct method.

Keywords: Block method; higher order ordinary differential equations; two point

ABSTRAK

Kaedah blok tersirat secara terus bagi dua-titik empat langkah yang berasaskan aplikasi kaedah Adams-Moulton yang ringkas untuk menyelesaikan secara terus sistem persamaan pembezaan biasa (PPB) am peringkat ketiga menggunakan saiz langkah yang berubah. Kaedah ini dilaksanakan bagi mendapatkan penyelesaian masalah nilai awal (MNA) pada dua titik secara serentak di dalam blok dengan menggunakan empat langkah sebelumnya. Hasil berangka menunjukkan bahawa kaedah blok yang dibangunkan adalah lebih baik daripada segi ralat maksimum pada semua toleran yang di uji dan kurang jumlah bilangan langkah apabila toleran semakin kecil jika dibandingkan dengan kaedah secara terus sedia ada.

Kata kunci: Kaedah blok; dua-titik; persamaan pembezaan biasa peringkat tinggi

INTRODUCTION

In this paper, we considered solving initial value problems (IVPs) for third order ordinary differential equations (ODEs) in the form:

$$y''' = f(x, y, y', y''),\ y(a) = \alpha,\ y'(a) = \beta,$$
$$y''(a) = \gamma,\ x \in [a,b] \qquad (1)$$

Equation (1) has been practically used in a wide variety of applications especially in science and engineering field and some other area of applications. The reduction of (1) to the system of first-order equations will leads to computation cost. The purpose of the present paper was to develop an alternative approach for the direct solution of (1) based on the direct variable step two-point four step (D2P4VS).

Several researchers such as Suleiman (1979), Lambert (1993), Omar (1999), Awoyemi (2003), Awoyemi and Idowu (2005), Majid and Suleiman (2006), Yap et al. (2008), Jator and Li (2009), Olabode and Yusuph (2009), and Majid et al. (2009, 2010) have investigated and suggested the best approach for solving the system of higher order ODEs directly. Majid and Suleiman (2006) proposed a one-point direct method for solving second order ODEs directly. The authors have shown that the computation of divided difference and integration coefficients in the code for the multistep method are very expensive. Yap et al. (2008) has introduced the two-point and three-point block method based on Newton-Gregory backward interpolation formula for solving special second order ODEs using constant step size while Majid et al. (2009) has developed a two-point block method in the form of Adams Moulton type for solving general second order ODEs directly using variable step size. Olabode and Yusuph (2009) has introduced a direct 6-steps block method for solving special third order ODEs using constant step size. A P-stable multistep method based on the collocation of the differential system from a basis function has been introduced by Awoyemi (2003) for solving general third order IVPs of ODEs directly using constant step size.

The essential aim of this paper was to propose a two-point four step direct implicit block method of order 7 for solving (1) using variable step size strategy. The propose block method in this paper will store all the coefficients of the method, therefore it manage to avoid the computation of the integration coefficients.

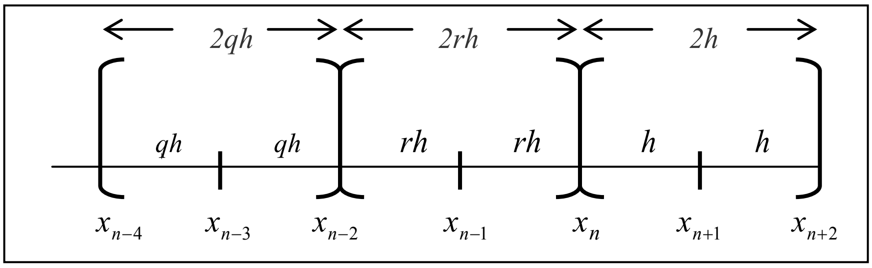

FIGURE 1 2-point 4 steps block method

FORMULATION OF THE METHOD

The interval [a, b] is divided into a series of blocks that involved the interpolation points from $(x_{n-4}, f_{n-4}),\ldots,$ (x_{n+2}, f_{n+2}) as shown in Figure 1. The solutions of y_{n+1} and y_{n+2} will be computed at several distinct points on the x-axis simultaneously in a block. In Figure 1, the computed block has the step size $2h$ while the previous block has the step size $2rh$ and $2qh$.

The general form of the k point formulation method can be written as follows:

$$y_{n+k}^{(3-p)} = \sum_{m=0}^{p-1} \frac{(kh)^m}{m!} y_n^{(3-p+m)} + h^p \sum_{j=0}^{s} \beta_{p,j}^{(k)} f_{n+2-j} \quad p = 1,2,3, \tag{2}$$

where

$$h^p \sum_{j=0}^{s} \beta_{p,j}^{(k)} f_{n+2-j} = \underbrace{\int_{x_n}^{x_{n+k}} \int_{x_n}^{x} \cdots \int_{x_n}^{x}}_{p\ times} P_{s,n+2}, \tag{3}$$

and $P_{s,n+2} \sum_{j=0}^{s} L_{s,j} f_{n+2-j}$

is the Lagrange polynomial of degree s.

The first point, y_{n+1}, can be obtained by taking $k = 1$and s = 6 in (2) and (3), hence the formulae of y_{n+1} in terms of r and q can be obtained by integrating (3) using MATHEMATICA which produces the following formulae*:*

Integrating once:

$$\begin{aligned} y''_{n+1} = y''_n + h \cdot \Bigg[& -\frac{(10 + 98r + 42q + 210rq^2 + 280r^2q^2 + 840r^3q + 42q^2 + 840r^2q + 315rq + 378r^2 + 700r^3 + 560r^4)}{3360(2 + r)(1 + r)(2 + 2r + q)(1 + r + q)} f_{n+2} \\ & + \frac{(40 + 343r + 147q + 525rq^2 + 560r^2q^2 + 1680r^3q + 126q^2 + 2100r^2q + 945rq + 1134r^2 + 1750r^3 + 1120r^4)}{210(1 + r)(1 + 2r)(1 + 2r + q)(1 + 2r + 2q)} f_{n+1} \\ & + \frac{(18 + 196r + 84q + 630rq^2 + 1400r^2q^2 + 4200r^3q + 98q^2 + 2520r^2q + 735rq + 882r^2 + 2100r^3 + 2800r^4)}{3360(r^2)(2r + q)(r + q)} f_n \\ & - \frac{(9 + 84r + 42q + 210rq^2 + 49q^2 + 630r^2q + 294rq + 294r^2 + 420r^3)}{210r^2(1 + r)(2 + r)(r + q)(r + 2q)} f_{n-1} \\ & + \frac{(18 + 140r + 84q + 210rq^2 + 98q^2 + 630r^2q + 441rq + 329r^2 + 420r^3)}{3360(r^2q^2)(1 + 2r)(1 + r)} f_{n-2} \\ & - \frac{(9 + 70r + 28q + 210r^2q + 147rq + 196r^2 + 210r^3)}{210(q^2)(r + q)(q + 2r)(1 + 2r + q)(2 + 2r + q)} f_{n-3} \\ & + \frac{(18 + 140r + 28q + 210r^2q + 147rq + 392r^2 + 420r^3)}{3360q^2(r + 2q)(r + q)(1 + 2r + 2q)(1 + r + q)} f_{n-4} \Bigg]. \end{aligned} \tag{4}$$

Integrating twice:

$$y'(x_{n+1}) = y'(x_n) + hy''(x_n) + h^2$$
$$\cdot\Bigg[-\frac{(5 + 56r + 24q + 168rq^2 + 280r^2q^2 + 840r^3q + 28q^2 + 672r^2q + 210rq + 252r^2 + 560r^3 + 560r^4)}{6720(2 + r)(1 + r)(2 + 2r + q)(1 + r + q)} f_{n+2}$$
$$+ \frac{(25 + 252r + 108q + 588rq^2 + 840r^2q^2 + 2520r^3q + 112q^2 + 2352r^2q + 840rq + 1008r^2 + 1960r^3 + 1680r^4)}{840(1 + r)(1 + 2r)(1 + 2r + q)(1 + 2r + 2q)} f_{n+1}$$
$$+ \frac{(11 + 140r + 60q + 672rq^2 + 1960r^2q^2 + 5880r^3q + 84q^2 + 2688r^2q + 630rq + 756r^2 + 2240r^3 + 3920r^4)}{6720(r^2)(2r + q)(r + q)} f_n$$
$$- \frac{(11 + 120r + 60q + 210rq^2 + 448q^2 + 1344r^2q + 504rq + 504r^2 + 896r^3)}{840r^2(1 + r)(2 + r)(r + q)(r + 2q)} f_{n-1}$$
$$+ \frac{(11 + 100r + 60q + 224rq^2 + 84q^2 + 672r^2q + 378rq + 336r^2 + 448r^3)}{6720(r^2q^2)(1 + 2r)(1 + r)} f_{n-2}$$
$$- \frac{(11 + 100r + 40q + 448r^2q + 252rq + 336r^2 + 448r^3)}{840(q^2)(r + q)(q + 2r)(1 + 2r + q)(2 + 2r + q)} f_{n-3}$$
$$+ \frac{(11 + 100r + 20q + 224r^2q + 126rq + 336r^2 + 448r^3)}{6720q^2(r + 2q)(r + q)(1 + 2r + 2q)(1 + r + q)} f_{n-4}\Bigg]. \quad (5)$$

Integrating three times:

$$y_{n+1} - y_n - hy'_n - \frac{h^2}{2!} y''_n = h^3$$
$$\cdot\Bigg[-\frac{(5 + 63r + 27q + 252rq^2 + 504r^2q^2 + 1512r^3q + 36q^2 + 1008r^2q + 270rq + 324r^2 + 840r^3 + 1008r^4)}{40320(2 + r)(1 + r)(2 + 2r + q)(1 + r + q)} f_{n+2}$$
$$+ \frac{(20 + 231r + 99q + 756rq^2 + 1344r^2q^2 + 4032r^3q + 120q^2 + 3024r^2q + 900rq + 1080r^2 + 2520r^3 + 2688r^4)}{5040(1 + r)(1 + 2r)(1 + 2r + q)(1 + 2r + 2q)} f_{n+1}$$
$$+ \frac{(13 + 189r + 81q + 1260rq^2 + 4536r^2q^2 + 13608r^3q + 132q^2 + 5040r^2q + 990rq + 1188r^2 + 4200r^3 + 9072r^4)}{40320(r^2)(2r + q)(r + q)} f_n$$
$$- \frac{(13 + 162r + 81q + 840rq^2 + 132q^2 + 2520r^2q + 792rq + 792r^2 + 1680r^3)}{5040(r^2)(1 + r)(2 + r)(r + q)(r + 2q)} f_{n-1}$$
$$+ \frac{(13 + 135r + 81q + 420rq^2 + 132q^2 + 1260r^2q + 594rq + 528r^2 + 840r^3)}{40320(r^2q^2)(1 + 2r)(1 + r)} f_{n-2}$$
$$- \frac{(13 + 135r + 54q + 840r^2q + 396rq + 528r^2 + 840r^3)}{5040(q^2)(r + q)(q + 2r)(1 + 2r + q)(2 + 2r + q)} f_{n-3}$$
$$+ \frac{(13 + 135r + 27q + 420r^2q + 198rq + 528r^2 + 840r^3)}{40320q^2(r + 2q)(r + q)(1 + 2r + 2q)(1 + r + q)} f_{n-4}\Bigg]. \quad (6)$$

The same process as obtaining the corrector for the first point is applied to derive the corrector formulae at the second point. The second point, y_{n+2}, can be obtained by taking $k = 2$ and $s = 6$ in (2) and (3). The formulae of y_{n+2} in terms of r and q can be obtained by integrating (3) using MATHEMATICA which produces the following formulae:

Integrate once:

$$y''(x_{n+2}) = y''(x_n) + h$$
$$\cdot\Bigg[\frac{(784r + 336q + 210rq^2 + 70r^2q^2 + 210r^3q + 126q^2 + 840r^2q + 945rq + 1134r^2 + 700r^3 + 140r^4 + 200)}{210(2+r)(1+r)(2+2r+q)(1+r+q)} f_{n+2}$$
$$+\frac{(320 + 1568r + 672q + 840rq^2 + 560r^2q^2 + 1680r^3q + 336q^2 + 3360r^2q + 2520rq + 3024r^2 + 2800r^3 + 1120r^4)}{105(1+r)(1+2r)(1+2r+q)(1+2r+2q)} f_{n+1}$$
$$+\frac{(-24 - 98r - 42q + 70r^2q^2 + 210r^3q - 14q^2 - 105rq - 126r^2 + 140r^4)}{210(r^2)(2r+q)(r+q)} f_n$$
$$+\frac{(96 + 192r + 168q + 56q^2 + 336rq + 336r^2 + 420r^3)}{105r^2(1+r)(2+r)(r+q)(r+2q)} f_{n-1} - \frac{(24 + 70r + 42q + 14q^2 + 63rq + 56r^2)}{210(r^2q^2)(1+2r)(1+r)} f_{n-2}$$
$$-\frac{(96 + 280r + 112q + 168rq + 224r^2)}{105(q^2)(r+q)(q+2r)(1+2r+q)(2+2r+q)} f_{n-3} - \frac{(24 + 70r + 14q + 21rq + 56r^2)}{210q^2(r+2q)(r+q)(1+2r+2q)(1+r+q)} f_{n-4}\Bigg]. \quad (7)$$

Integrate twice:

$$y'(x_{n+2}) - y'(x_n) - 2hy''(x_n) = h^2$$
$$\cdot\Bigg[\frac{(20 + 84r + 36q + 21rq^2 + 14q^2 + 84r^2q + 105rq + 126r^2 + 70r^3)}{105(2+r)(1+r)(2+2r+q)(1+r+q)} f_{n+2}$$
$$+\frac{(160 + 896r + 384q + 672rq^2 + 560r^2q^2 + 1680r^3q + 224q^2 + 2688r^2q + 1680rq + 2016r^2 + 2240r^3 + 1120r^4)}{105(1+r)(1+2r)(1+2r+q)(1+2r+2q)} f_{n+1}$$
$$+\frac{(-4 - 14r - 6q + 21rq^2 + 70r^2q^2 + 210r^3q + 84r^2q + 70r^3 + 140r^4)}{105(r^2)(2r+q)(r+q)} f_n$$
$$-\frac{(-32 - 96r - 48q + 112rq^2 + 336r^2q + 224r^3)}{105r^2(1+r)(2+r)(r+q)(r+2q)} f_{n-1} + \frac{(-4 - 10r - 6q - 7rq^2 + 21r^2q + 14r^3)}{105(r^2q^2)(1+2r)(1+r)} f_{n-2}$$
$$-\frac{(-32 - 80r - 32q + 112r^2q + 112r^3)}{105(q^2)(r+q)(q+2r)(1+2r+q)(2+2r+q)} f_{n-3} + \frac{(-4 - 10r - 2q + 7r^2q + 14r^3)}{105q^2(r+2q)(r+q)(1+2r+2q)(1+r+q)} f_{n-4}\Bigg]. \quad (8)$$

Integrate thrice:

$$y(x_{n+2}) - y(x_n) - 2hy'(x_n) - \frac{(2h)^2}{2!} y''(x_n) = \frac{h^3}{2}$$
$$\cdot\Bigg[-\frac{\left(-10 - 42r - 18q + 21r^2q^2 - 6q^2 + 63r^3q - 45rq - 54r^2 + 42r^4\right)}{315(2+r)(1+r)(2+2r+q)(1+r+q)} f_{n+2}$$
$$+\frac{\left(160 + 1008r + 432q + 1008rq^2 + 1008r^2q^2 + 3024r^3q + 288q^2 + 4032r^2q + 2160rq + 2592r^2 + 3360r^3 + 2016r^4\right)}{315(1+r)(1+2r)(1+2r+q)(1+2r+2q)} f_{n+1}$$
$$+\frac{\left(567r^3q + 189r^2q^2 + 54r^2 + 45rq + 252r^2q + 6q^2 + 63rq^2 + 210r^3 + 378r^4 - 2\right)}{315(r^2)(2r+q)(r+q)} f_n$$
$$-\frac{\left(-16 + 48q^2 + 336rq^2 + 1008r^2q + 672r^3 + 288r^2 + 288rq\right)}{315(r^2)(1+r)(2+r)(r+q)(r+2q)} f_{n-1} + \frac{\left(-2 + 6q^2 + 21rq^2 + 63r^2q + 42r^3 + 24r^2 + 27rq\right)}{315(r^2q^2)(1+2r)(1+r)} f_{n-2}$$
$$-\frac{\left(-16 + 336r^2q - 336r^3 + 192r^2 + 144rq\right)}{315(q^2)(r+q)(q+2r)(1+2r+q)(2+2r+q)} f_{n-3} + \frac{\left(-2 + 9rq + 21r^2q + 42r^3 + 24r^2\right)}{315(q^2)(r+2q)(r+q)(1+2r+2q)(1+r+q)} f_{n-4}\Bigg]. \quad (9)$$

The two-point four step implicit block method is the combination of predictor of order 6 and the corrector of order 7. The formulae for predictor can be derived in a similar way as the corrector, but the interpolation points involved are $(x_{n-5}, f_{n-5}),\ldots,(x_n, f_n)$.

IMPLEMENTATION OF THE METHOD

During the implementation of the method, the choices of the next step size is restricted to half, double or constant. The successful step size remains constant for at least two blocks before allowing it to be doubled. In case of successful step size, if the step size remain the same then the ratios are (r=1, q=1), (r=1, q=2) or (r=1, q=0.5). When the step size is doubled, the ratios is (r=0.5, q=0.5). In case of step size failure, the choices of ratios is (r=2, q=2). For instance, taking (r=1.0, q=1.0) in (4), (5), (6), (7), (8) and (9), we obtained the corrector formulae of the first point and second point as follows:

First point:

$$y''_{n+1} = y''_n + \frac{h}{60480}\left(-863 f_{n+2} + 25128 f_{n+1} + 46989 f_n - 16256 f_{n+1} + 7299 f_{n-2} - 2088 f_{n-3} + 271 f_{n-4}\right).$$

$$y'_{n+1} = y'_n + h y''_n + \frac{h^2}{120960}\left(-731 f_{n+2} + 14742 f_{n+1} + 57123 f_n - 15884 f_{n-1} + 6939 f_{n-2} - 1962 f_{n-3} + 253 f_{n-4}\right).$$

$$y_{n+1} = y_n + h y'_n + \frac{h^2}{2!} y''_n + \frac{h^3}{3628800}\left(-5849 f_{n+2} + 100884 f_{n+1} + 604635 f_n - 140240 f_{n-1} + 60045 f_{n-2} - 16836 f_{n-3} + 2161 f_{n-4}\right). \tag{10}$$

Second point:

$$y''_{n+2} = y''_n + \frac{h}{3780}\left(1139 f_{n+2} + 5640 f_{n+1} + 33 f_n + 1328 f_{n-1} - 807 f_{n-2} + 264 f_{n-3} - 37 f_{n-4}\right).$$

$$y'_{n+2} = y'_n + 2h y''_n + \frac{h^2}{1890}\left(112 f_{n+2} + 2184 f_{n+1} + 1713 f_n - 248 f_{n-1} + 66 f_{n-2} - 12 f_{n-3} + f_{n-4}\right).$$

$$y_{n+2} = y_n + 2h y'_n + \frac{(2h)^2}{2!} y''_n + \frac{h^3}{28350}\left(49 f_{n+2} + 15816 f_{n+1} + 26430 f_n - 6560 f_{n-1} + 2715 f_{n-2} - 744 f_{n-3} + 94 f_{n-4}\right). \tag{11}$$

In the code, the values of y_{n+1} and y_{n+2} were approximated using the predictor-corrector schemes. If t corrections are needed, then the sequence of computations at any mesh point is $(PE)(CE)^t$ where P and C indicate the application of the predictor and corrector formulae respectively and E indicate the evaluation of the function f. A simple iteration has been implemented to approximate the values of y_{n+1} and y_{n+2}. In the code, we iterate the corrector to convergent and the convergence test employed was:

$$\left| y^{(t)}_{n+2} - y^{(t-1)}_{n+2} \right| < 0.1 \times TOL$$

where t is the number of iterations. After the successful convergence test, local errors estimate *Est* at the point x_{n+2} are performed to control the error for the block. We obtained the *Est* by comparing the absolute difference of the corrector formula derived of order k and a similar corrector formula of order *(k-1)*. The error controls for the developed methods are at the second point in the block because generally it had given us better results.

The errors calculated in the code are defined as:

$$(E_i)_t = \left| \frac{(y_i)_t - (y(x_i))_t}{A + B(y(x_i))_t} \right|,$$

where $(y)_t$ is the t-th component of the approximate y. A=1, B=0 correspond to the absolute error test. A=1, B=1 correspond to the mixed test and finally A=0, B=1 correspond to the relative error test. The mixed error test is used for Problem 1, 2, 3 and 4. The maximum error is defined as follows:

$$MAXE = \max_{1 \le i \le SSTEP} \left(\max_{1 \le i \le N} (E_i)_t \right),$$

where N is the number of equations in the system and SSTEP is the number of successful steps. At each step of integration, a test for checking the end of the interval is made. If b denotes the end of the interval then:

$$\text{if } x+2h \geq b \quad \text{then } h = \frac{b-x}{2},$$

otherwise h remain as calculated. The technique above helped to reach the end point of the interval. The code was written in C language and executed on UNIX operating system.

STABILITY ANALYSIS

In this section, we will discuss the stability of the proposed method derived in the previous section on a linear third order problem,

$$y''' = f = \beta y'' + \theta y' + \lambda y \tag{12}$$

Applying equation (12) into the corrector formula of y_{n+1} and y_{n+2} in (10) and (11). Then, the formulae are written into matrix form and setting the determinant of the matrix to zero. Hence, the stability polynomial is obtained,

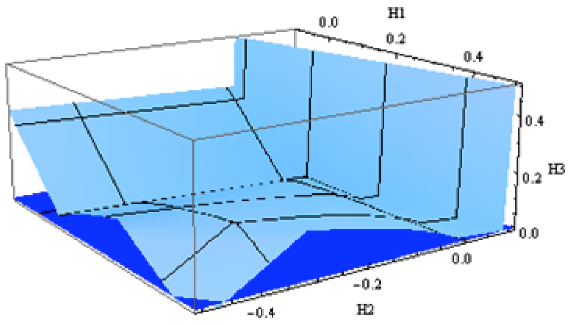

FIGURE 2 Stability region for D2P4VS when $r = q = 1$.

where $H1 = h\phi$, $H2 = h^2\lambda$ and $H3 = h^3\beta$. Figure 2 show the stability region of the D2P4VS method when $r = q = 1$. The stability region is plotted using MATHEMATICA and the shaded region in Figure 2 demonstrate the stability region for the proposed method when $r = q = 1$.

$$\begin{aligned}
Q(H1,H2,H3) = &\left(1 - \frac{5419}{7560}H1 + \frac{99667}{680400}H1^2 - \frac{313}{1728}H2 + \frac{134647}{1555200}H2H1 + \frac{268441}{19051200}H2^2 - \frac{24023}{3402000}H3H2 + \frac{12031}{12700800}H3^2\right)t^{18} \\
&+ \left(-3 + \frac{311}{504}H1 - \frac{207461}{1360780}H1^2 - \frac{30431}{8640}H2 - \frac{1781359}{1555200}H2H1 - \frac{1702531}{2721600}H2^2 - \frac{171433}{43200}H3 + \frac{45795497}{163296000}H3H1 - \frac{23031611}{47628000}H3H2 \right. \\
&\left. - \frac{29473403}{190512000}H3^2\right)t^{17} + \left(3 + \frac{2057}{3780}H1 + \frac{155447}{75600}H1^2 + \frac{21853}{6048}H2 - \frac{23801}{34560}H2H1 + \frac{192341}{226800}H2^2 - \frac{1776029}{453600}H3 - \frac{81804239}{18144000}H3H1 \right. \\
&\left. - \frac{29349217}{95256000}H3H2 + \frac{119384509}{19051200}H3^2\right)t^{16} + \left(-1 - \frac{631}{3780}H1 - \frac{23809}{136080}H1^2 + \frac{3151}{30240}H2 + \frac{2473007}{2177280}H2H1 - \frac{145}{1944}H2^2 - \frac{67289}{453600}H3 \right. \\
&\left. - \frac{72617551}{32659200}H3H1 + \frac{1804283}{6350400}H3H2 + \frac{7757621}{38102400}H3^2\right)t^{15} + \left(-\frac{461}{2520}H1 - \frac{61619}{136080}H1^2 - \frac{3271}{60480}H2 + \frac{246203}{435456}H2H1 - \frac{81929}{544320}H2^2 \right. \\
&\left. + \frac{19349}{302400}H3 - \frac{10148069}{32659200}H3H1 + \frac{225853}{1905120}H3H2 - \frac{16669}{7620480}H3^2\right)t^{14} + \left(-\frac{143}{1512}H1 - \frac{3803}{75600}H1^2 + \frac{2411}{60480}H2 + \frac{56029}{1209600}H2H1 - \right. \\
&\frac{10421}{907200}H2^2 - \frac{2129}{907200}H3 - \frac{32923}{2592000}H3H1 + \frac{175129}{23814000}H3H2 - \left. \frac{48821}{38102400}H3^2\right)t^{13} + \left(-\frac{1}{19440}H1^2 + \frac{223}{10886400}H2H1 - \frac{1}{194400}H2^2 \right. \\
&+ \frac{1429}{23328000}H3H1 - \frac{653}{95256000}H3H2 - \left. \frac{1177}{190512000}H3^2\right)t^{12} + \frac{17}{680400}H1^2 - \frac{59}{2177280}H2H1 + \frac{43}{9525600}H2^2 + \frac{12977}{1143072000}H3H1 \\
&- \frac{163}{95256000}H3H2 - \frac{23}{63504000}H3^2 \Big)t^{11}
\end{aligned}$$

RESULTS AND DISCUSSION

In order to study the efficiency of the developed code, we presented four numerical experiments for the following test problems. All the four tested problems are in Awoyemi (2003). Problem 3 and 4 also can be found in Awoyemi and Idowu (2005).

Problem 1:

$$y''' + 4y' = x,$$
$$y(0) = y'(0) = 0, y''(0) = 1, \quad 0 \leq x \leq b$$

The exact solution: $y(x) = (3/16)(1-\cos 2x) + (1/8)x^2$.

Problem 2:

$$y''' + y' = 0,$$
$$y(0) = 0, y'(0) = 1, y''(0) = 2, \quad 0 \leq x \leq b$$

The exact solution: $y(x) = 2(1-\cos x) + \sin x$.

Problem 3:

$$y''' + 2y'' - 9y' - 18y = -18x^2 - 18x + 22,$$
$$y(0) = -2, y'(0) = -8, y''(0) = -12, 0 \leq x \leq b$$

The exact solution: $y(x) = -2e^{-3x} + e^{-2x} + x^2 - 1$.

Problem 4:

$$y''' - 2y'' - 3y' + 10y = 34xe^{-2x} - 16e^{-2x} - 10x^2 + 6x + 34,$$
$$y(0) = 3, y'(0) = 0, y''(0) = 0$$
$$0 \le x \le b$$

The exact solution: $y(x) = x^2 e^{-2x} - x^2 + 3$.

The notations used in the Table 1 – 5 are as follows:

TOL	Tolerance
MTD	Method employed
b	End of interval
TS	Total steps taken
MAXE	Magnitude of the maximum error of the computed solution
FCN	Total function calls
D2P4VS	Implementation of the direct two-point four step implicit block method derived earlier using variable step size
Awoyemi(1)	Numerical results in Awoyemi (2003)
Awoyemi(2)	Numerical results in Awoyemi and Idowu (2005)

The codes are written in C language and executed on DYNIX/ptx operating system. The total number of steps and maximum error between D2P4VS, Awoyemi(1) and Awoyemi(2) are presented in Figures 3 to 6 and in Table 1 to 4 for solving problem 1 to 4.

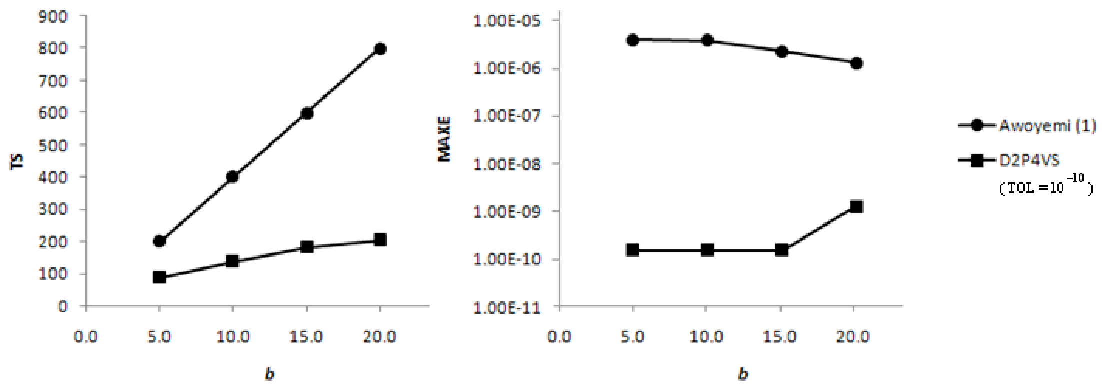

FIGURE 3. Results of total steps and maximum error for Problem 1

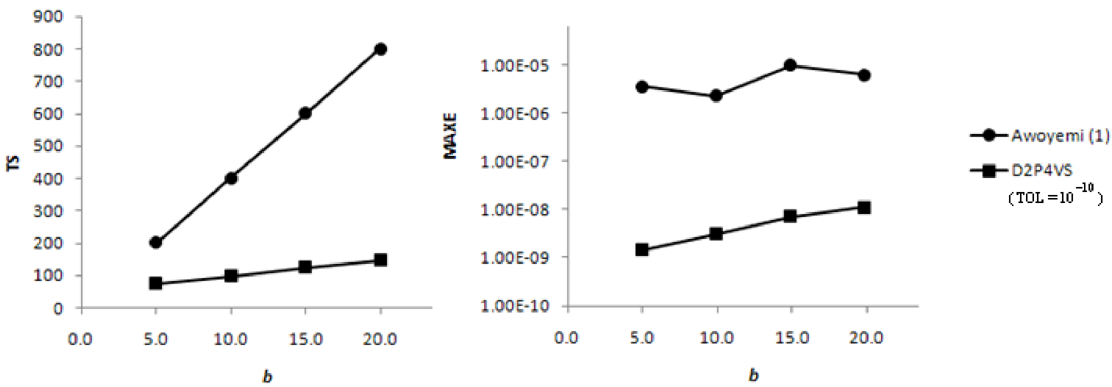

FIGURE 4. Results of total steps and maximum error for Problem 2

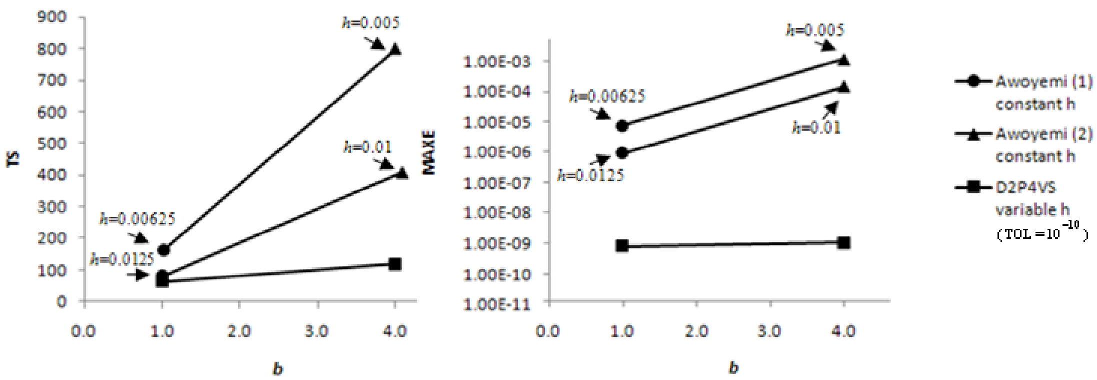

FIGURE 5. Results of total steps and maximum error for Problem 3

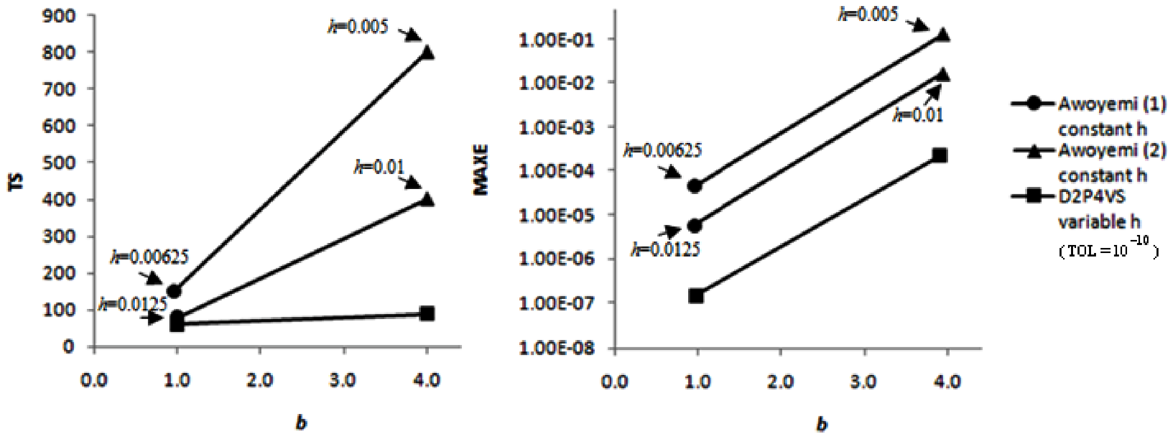

FIGURE 6: Results of total steps and maximum error for Problem 4

In Figure 3 – 6, it is obvious that method D2P4VS requires less number of total steps as compared to method Awoyemi(1) and Awoyemi(2) when solving the same given problems. It is also observed that the maximum error of D2P4VS at tolerance 10^{-10} are smaller than Awoyemi(1) and Awoyemi(2) at all different values of b.

TABLE 1: Comparison results for solving Problem 1

	Awoyemi(1)			D2P4VS				
Step size	*b*	TS	MAXE	*b*	TOL	TS	MAXE	FCN
0.025	5.0	200	3.94(-6)	5.0	10^{-6}	46	4.66(-7)	242
					10^{-8}	56	9.14(-8)	306
					10^{-10}	88	1.53(-10)	468
	10.0	400	3.80(-6)	10.0	10^{-6}	61	4.66(-7)	328
					10^{-8}	91	2.43(-8)	436
					10^{-10}	136	1.53(-10)	722
	15.0	600	2.29(-6)	15.0	10^{-6}	76	4.66(-7)	406
					10^{-8}	110	2.63(-8)	550
					10^{-10}	180	1.54(-10)	918
	20.0	800	1.30(-6)	20.0	10^{-6}	91	4.66(-7)	478
					10^{-8}	129	2.63(-8)	666
					10^{-10}	204	1.28(-9)	1062

TABLE 2: Comparison results for solving Problem 2

	Awoyemi(1)			D2P4VS				
Step size	*b*	TS	MAXE	*b*	TOL	TS	MAXE	FCN
0.025	5.0	200	3.53(-6)	5.0	10^{-6}	43	1.86(-7)	214
					10^{-8}	56	2.30(-9)	284
					10^{-10}	75	1.39(-9)	364
	10.0	400	2.25(-6)	10.0	10^{-6}	50	2.65(-6)	260
					10^{-8}	75	1.29(-8)	390
					10^{-10}	99	3.01(-9)	508
	15.0	600	9.85(-6)	15.0	10^{-6}	58	1.11(-5)	316
					10^{-8}	94	2.82(-8)	502
					10^{-10}	123	6.96(-9)	652
	20.0	800	6.31(-6)	20.0	10^{-6}	66	2.64(-5)	372
					10^{-8}	113	4.38(-8)	608
					10^{-10}	146	1.08(-8)	792

TABLE 3. Comparison results for solving Problem 3

Step size	Awoyemi(1) b	TS	MAXE	D2P4VS b	TOL	TS	MAXE	FCN
0.0125	1.0	80	7.60(-6)	1.0	10^{-6}	41	9.33(-7)	210
					10^{-8}	54	7.82(-8)	256
0.00625		160	9.54(-7)		10^{-10}	64	8.16(-10)	330
Step size	**Awoyemi(2) b**	**TS**	**MAXE**	**D2P4VS**	**TOL**	**TS**	**MAXE**	**FCN**
0.01	4.0	400	1.16(-3)	4.0	10^{-6}	59	2.26(-6)	318
					10^{-8}	99	7.82(-8)	436
0.005		800	1.46(-4)		10^{-10}	120	1.07(-9)	666

TABLE 4. Comparison results for solving Problem 4

Step size	Awoyemi(1) b	TS	MAXE	D2P4VS b	TOL	TS	MAXE	FCN
0.0125	1.0	80	4.90(-5)	1.0	10^{-6}	38	4.36(-5)	194
					10^{-8}	48	2.32(-6)	234
0.00625		160	6.20(-6)		10^{-10}	60	1.48(-7)	296
Step size	**Awoyemi(2) b**	**TS**	**MAXE**	**D2P4VS**	**TOL**	**TS**	**MAXE**	**FCN**
0.01	4.0	400	1.18(-1)	4.0	10^{-6}	49	5.11(-3)	260
					10^{-8}	65	5.08(-4)	346
0.005		800	1.48(-2)		10^{-10}	89	2.18(-4)	450

Tables 1 and 2 show that D2P4VS managed to obtain better accuracy and less total number of steps compared to Awoyemi(1) when b = 5.0, 10.0, 15.0 and 20.0. Concerning Table 1, when b = 5.0, in Awoyemi (1) the maximum error was 3.94(-6) with 200 steps and when b = 20.0 the maximum error was 1.30(-6) using 800 steps. While the D2P4VS could obtain the maximum error of 1.53(-10) (when b = 5.0) and 1.28(-9) (when b = 20.0) with 88 and 204 steps respectively at TOL = 10^{-10}. The same pattern can be observed in Table 2.

In Table 3, when b = 1.0, in Awoyemi(1) the best results was achieved with 160 steps and the maximum error was 9.54(-7) and when b = 4.0, in Awoyemi(2) the optimum accuracy was 1.46 (-4) using 800 steps. While D2P4VS could obtain the maximum error of 8.16(-10) (when b = 1.0) and 1.07(-9) (when b = 4.0) using 64 and 120 steps, respectively. Table 3 also shows the advantage of D2P4VS over Awoyemi(1) and Awoyemi(2). Hence, the method proposed is clearly superior since it involves less computational cost and obtained highly accurate results.

CONCLUSION

In this paper, we have shown the efficiency of the developed two-point four step block method presented in the simple form of Adams-Moulton method using variable step size was suitable for solving general third-order ODEs. The method has shown the superiority in terms of total steps, function calls and maximum error over the existence method in Awoyemi (2003) and Awoyemi and Idowu (2005).

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