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On ξ^a -Quadratic Stochastic Operators on 2-D Simplex

(ξ^a -Quadratik Stochastic Pengendali Di Simplex 2-D)

FARRUKH MUKHAMEDOV*, IZZAT QARALLEH & WAN NUR FAIRUZ ALWANI BT WAN ROZALI

ABSTRACT

A quadratic stochastic operator (QSO) is usually used to present the time evolution of differing species in biology. Some quadratic stochastic operators have been studied by Lotka and Volterra. The general problem in the nonlinear operator theory is to study the behavior of operators. This problem was not fully finished even for quadratic stochastic operators which are the simplest nonlinear operators. To study this problem, several classes of QSO were investigated. In this paper, we study the $\xi^{(a)}$–QSO defined on 2D simplex. We first classify $\xi^{(a)}$–QSO into 2 non-conjugate classes. Further, we investigate the dynamics of these classes of such operators.

Keywords: Fixed point; quadratic stochastic operator

ABSTRAK

Pengendali stokastik kuadratik (QSO) biasanya digunakan untuk menunjukkan evolusi masa berbeza spesies dalam biologi. Sesetengah pengendali stokastik kuadratik telah dikaji oleh Lotka dan Volterra. Masalah umum dalam teori tak linear pengendali adalah untuk mengkaji tingkah laku pembekal. Masalah ini tidak sepenuhnya siap untuk pengendali stokastik kuadratik yang merupakan pengendali tak linear yang paling mudah. Untuk memahami masalah ini, beberapa kelas QSO telah dikaji. Dalam kertas ini, kami mengkaji $\xi^{(a)}$– QSO yang ditentukan pada simpleks 2D. Kami mengklasifikasikan $\xi^{(a)}$– QSO ke dalam kelas bukan konjugat. Seterusnya, kami mengkaji kedinamikan kelas pengusaha terbabit.

Kata kunci: Pengendali stokastik kuadratik; titik tetap

INTRODUCTION

The history of quadratic stochastic operators can be traced back to Bernstein's work (Bernstein 1924). The quadratic stochastic operator was considered an important source of analysis for the study of dynamical properties and modelings in various fields such as biology (Bernstein 1942; Hofbauer & Sigmund 1988; Hofbauer et al. 1987; Li et al. 2006; Lotka 1920; Lyubich Yu 1992; Volterra 1927), physics (Plank & Losert 1995; Udwadia & Raju 1998), economics and mathematics (Hofbauer & Sigmund 1988; Kesten 1970; Lyubich Yu 1992; Ulam 1964).

One of such systems which relates to the population genetics is given by a quadratic stochastic operator (Bernstein 1942). A quadratic stochastic operator (in short QSO) is usually used to present the time evolution of species in biology, which arises as follows. Consider a population consisting of m species (or traits) $1, 2, \ldots, m$. We denote a set of all species (traits) by $I = \{1, 2, \ldots, m\}$. Let $x^{(0)} = (x_1^{(0)}, \ldots, x_m^{(0)})$ be a probability distribution of species at an initial state and $P_{ij,k}$ be a probability that individuals in the i^{th} and j^{th} species (traits) interbreed to produce an individual from k^{th} species (trait). Then a probability distribution $x^{(1)} = (x_1^{(1)}, \ldots, x_m^{(1)})$ of the spices (traits) in the first generation can be found as a total probability, i.e.,

$$x_k^{(1)} = \sum_{i,j=1}^{m} P_{ij,k} x_i^{(0)} x_j^{(0)}, \quad k = \overline{1,m}. \tag{1}$$

This means that the association $x^{(0)} \to x^{(1)}$ defines a mapping V called *the evolution operator*. The population evolves by starting from an arbitrary state $x^{(0)}$, then passing to the state $x^{(1)} = V(x^{(0)})$ (the first generation) and then to the state $x^{(2)} = V(x^{(1)}) = V(V(x^{(0)}) = V^{(2)}(x^{(0)})$ (the second generation). Therefore, the evolution states of the population system described by the following discrete dynamical system,

$$x^{(0)}, x^{(1)} = V(x^{(0)}), x^{(2)} = V^{(2)}(x^{(0)}), x^{(3)} = V^{(3)}(x^{(0)}) \ldots.$$

In other words, a QSO describes a distribution of the next generation if the distribution of the current generation was given. The fascinating applications of QSO to population genetics were given in Lyubich Yu (1992). We should stress that the mapping V defined by (1) is non-linear (quadratic) and therefore, to study the dynamics of V, it requires higher dimensional dynamical systems methods. On the other hand, higher-dimensional dynamical systems are important but there are relatively few dynamical phenomena that are currently understood. In Ganikhodzhaev et al. (2011) it was given along self-contained exposition of the recent achievements and open problems in the theory of the QSO.

The main problem in the nonlinear operator theory is to study the behavior of nonlinear operators. This problem was not fully finished even in the class of QSO (the QSO is the simplest nonlinear operator). The difficulty of the problem depends on the given cubic matrix $\left(P_{ijk}\right)_{i,j,k=1}^{m}$. An asymptotic behavior of the QSO even on the small dimensional simplex is complicated (Stein & Ulam 1962; Ulam 1964; Zakharevich 1978). In order to solve this problem, many researchers always introduced a certain class of QSO and studied their behavior (see for example (Ganikhodzhaev 1994; Jenks 1969; Mukhamedov & Saburov 2010; Mukhamedov et al. 2013; Rozikov & Zada 2010; Ulam 1964). However, all these classes together would not cover the set of all QSO. Therefore, there are many classes of QSO which were not studied yet. In this paper we are going to introduce a new class of QSO which is called a $\xi^{(a)}$-QSO. This class of operators depends on a partition of the coupled index set (the coupled trait set) $\mathbf{P}_m = \{(i, j) : i < j\} \subset I \times I$. In case of two dimensional simplex ($m = 3$), the coupled index set (the coupled trait set) $\mathbf{P}_3$ has five possible partitions.

In this present paper we are going to investigate $\xi^{(a)}$-QSO corresponding to the point partition (the maximal partition) of $\mathbf{P}_3$. To study such operators we first classify them into two non-conjugate classes. Further, we will investigate the dynamics of each class of such operators.

Preliminaries

Recall that a quadratic stochastic operator (QSO) is a mapping of the simplex

$$S^{m-1} = \left\{ x = (x_1, \ldots, x_m) \in R^m : \sum_{i=1}^{m} x_i = 1, x_i \geq 0, i = \overline{1,m} \right\}, \tag{2}$$

into itself, of the form

$$x'_k = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j, \; k = \overline{1,m}, \tag{3}$$

where $V(x) = x' = (x'_1, \ldots, x'_m)$ and $P_{ij,k}$ is a coefficient of heredity, which satisfies the following conditions

$$P_{ij,k} \geq 0, \; P_{ij,k} = P_{ji,k}, \; \sum_{k=1}^{m} P_{ij,k} = 1. \tag{4}$$

Thus, each quadratic stochastic operator $V : S^{m-1} \to S^{m-1}$ can be uniquely defined by a cubic matrix $P = \left(P_{ijk}\right)_{i,j,k=1}^{m}$ with conditions (4).

We denote sets of fixed points an k–periodic points of $V : S^{m-1} \to S^{m-1}$ by $Fix(V)$ and $Per_k(V)$, respectively. Due to Brouwer's fixed point theorem, one always has that $Fix(V) \neq \emptyset$ for any QSO V. For a given point $x^{(0)} \in S^{m-1}$, a trajectory $\left\{x^{(n)}\right\}_{n=0}^{\infty}$ of $V : S^{m-1} \to S^{m-1}$ starting from $x^{(0)}$ is defined by $x^{(n+1)} = V(x^{(n)})$. By $\omega_V(x^{(0)})$, we denote a set of omega limiting points of the trajectory $\left\{x^{(n)}\right\}_{n=0}^{\infty}$. Since $\left\{x^{(n)}\right\}_{n=0}^{\infty} \subset S^{m-1}$ and S^{m-1} is compact, one has that $\omega_V(x^{(0)}) \neq \emptyset$. Obviously, if $\omega_V(x^{(0)})$ consists of a single point, then the trajectory converges and a limiting point is a fixed point of V.

Recall that a continuous function $\varphi : S^{m-1} \to R$ is called a *Lyapunov function* for the dynamical system (3) if the limit $\lim_{n\to\infty} \varphi(V^\wedge(n))$ exists for any initial point x^0.

Note that each element $x \in S^{m-1}$ is a probability distribution of the set $I = \{1, \ldots, m\}$. Let $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$ be vectors taken from S^{m-1}. We say that x *is equivalent to* y if $x_k = 0 \Leftrightarrow y_k = 0$. We denote this relation by $x \sim y$.

Let $supp(x) = \{i : x_i \neq 0\}$ be a support of $x \in S^{m-1}$. We say that x *is singular to* y and denote by $x \perp y$, if $supp(x) \cap supp(y) = \emptyset$. Note that if $x, y \in S^{m-1}$ then $x \perp y$ if and only if $(x, y) = 0$, her $(\cdot, \cdot)$ stands for a standard inner product in R^m.

We denote sets of coupled indexes by

$$\mathbf{P}_m = \{(i, j) : i < j\} \subset I \times I, \Delta_m = \{(i, i) : i \in I\} \subset I \times I.$$

For a given pair $(i, j) \in \mathbf{P}_m \cup \Delta_m$, we set a vector $\mathrm{P}_{ij} = (P_{ij,1}, \ldots, P_{ij,m})$. It is clear due to the condition (4) that $\mathrm{P}_{ij} \in S^{m-1}$.

Let $\xi = \{A_i\}_{i=1}^{N}$ be some fixed partition of $\mathbf{P}_m$, i.e. $A_i \cap A_j = \emptyset$ and $\bigcup_{i=1}^{N} A_i = \mathbf{P}_m$ where $N \leq m$.

Definition 1. An operator $V : S^{m-1} \to S^{m-1}$ given by (3), (4) is called a *$\xi^{(a)}$-quadratic stochastic operator (QSO)* if the following conditions are satisfied:

(i) For any $(i, j), (u, v) \in A_k$, one has that $\mathrm{P}_{ij} \sim \mathrm{P}_{uv}$;

(ii) For any $(i, j) \in A_k$ and $(u, v) \in A_l$, where $k \neq l$, one has that $\mathrm{P}_{ij} \perp \mathrm{P}_{uv}$ and

(iii) For any $(i, i), (u, u) \in \Delta_m$, where $i \neq u$, one has that $\mathrm{P}_{ii} \sim \mathrm{P}_{uu}$.

Remark. We note that if one changes condition (iii), i.e. $\mathrm{P}_{ii} \sim \mathrm{P}_{uu}$ to $\mathrm{P}_{ii} \perp \mathrm{P}_{uu}$, then we get another class of QSO, which is called $\xi^{(s)}$-QSO. Such kind of operators have been investigated in (Mukhamedov & Jamal 2010; Mukhamedov & Saburov 2010; Mukhamedov et al. 2014, 2012). Going forward, we can say that dynamics of these two classes is totaly different.

A biological interpretation of a $\xi^{(a)}$-QSO: We treat $I = \{1, \ldots, m\}$ as a set of all possible traits of the population system. A coefficient $P_{ij,k}$ is a probability that parents in the i^{th} and j^{th} traits interbreed to produce a child from the k^{th} trait. The condition $P_{ij,k} = P_{ji,k}$ means that the gender of parents do not influence to have a child from the k^{th} trait. In this sense, $\mathbf{P}_m \cup \Delta_m$ is a set of all possible coupled traits of parents. A vector $\mathrm{P}_{ij} = (P_{ij,1}, \ldots, P_{ij,m})$ is a possible distribution of children in a family while parents are carrying traits from the i^{th} and j^{th} types. A biological meaning of a $\xi^{(a)}$-QSO is as follows: a set $\mathbf{P}_m$ of all differently coupled traits of parents is splitted into N groups $A_1, \ldots, A_N$ (here N is less than the number m of traits) such that the chance (probability) of having a child from any trait in two different family whose parents' coupled traits belong to the same group A_k is simultaneously either positive or zero (the condition (i) of Definition 1), meanwhile, two family whose parents' coupled traits belong to two different groups A_k and A_l cannot have a child from the same trait, simultaneously (the condition (ii) of Definition

1). Moreover, the parents which are sharing the same type of traits the chance (probability) of having a child from any trait simultaneously either positive or zero (the condition (iii) of Definition 1).

Descriptions of The Operators

In this section, we are going to study $\xi^{(a)}$-QSO in two dimensional simplex, i.e. $m = 3$. In this case, we have the following possible partitions of $\mathbf{P}_3$

$$\xi_1 = \{\{(1, 2)\}, \{(1, 3)\}, \{(2, 3)\}, |\xi_1| = 3,$$
$$\xi_2 = \{\{(2, 3)\}, \{(1, 2)\}, \{(1, 3)\}, |\xi_2| = 2,$$
$$\xi_3 = \{\{(1, 3)\}, \{(1, 2)\}, \{(2, 3)\}, |\xi_3| = 2,$$
$$\xi_4 = \{\{(1, 2)\}, \{(1, 3)\}, \{(2, 3)\}, |\xi_4| = 2,$$
$$\xi_5 = \{\{(1, 2), (1, 3), (2, 3)\}, |\xi_5| = 1.$$

In the present paper, we are aiming to study $\xi^{(a)}$-QSO related to the partition ξ_1. Other partitions will be studied elsewhere.

Let us recall that two operators V_1, V_2 are called *(topologically or linearly) conjugate*, if there is a permutation matrix P such that $P^{-1}V_1P = V_2$. Let π be a permutation of the set $I = \{1, \dots, m\}$. For any vector x, we define $\pi(x) = (x_{\pi(1)}, \dots, x_{\pi(m)})$. It is easy to check that if π is a permutation of the set I corresponding to the given permutation matrix P then one has that $Px = \pi(x)$. Therefore, two operators V_1, V_2 are conjugate if and only if $\pi^{-1}V_1\pi = V_2$ for some permutation π. Throughout this paper, we shall consider 'conjugate operators' in this sense. We say that two classes K_1 and K_2 of operators are conjugate if every operator taken from K_1 is conjugate to some operator taken from K_2 and vise versus and we denote it as $K_1 \simeq K_2$. Now, we shall consider some sub-class of a class of all $\xi^{(a)}$-QSO corresponding to the partition ξ_1 by choosing coefficients $\left(P_{ij,k}\right)_{i,j,k=1}^{3}$ in special forms:

Case	P_{11}	P_{22}	P_{33}
$\mathbf{I}_1$	$(a, b, 0)$	$(a, b, 0)$	$(a, b, 0)$
$\mathbf{I}_2$	$(b, a, 0)$	$(b, a, 0)$	$(b, a, 0)$
$\mathbf{I}_3$	$(0, a, b)$	$(0, a, b)$	$(0, a, b)$
$\mathbf{I}_4$	$(0, b, a)$	$(0, b, a)$	$(0, b, a)$
$\mathbf{I}_5$	$(a, 0, b)$	$(a, 0, b)$	$(a, 0, b)$
$\mathbf{I}_6$	$(b, 0, a)$	$(b, 0, a)$	$(b, 0, a)$

where $a, b \in [0, 1]$ and

Case	P_{12}	P_{13}	P_{23}
$\mathbf{II}_1$	(1,0,0)	(0,1,0)	(0,0,1)
$\mathbf{II}_2$	(0,1,0)	(1,0,0)	(0,0,1)
$\mathbf{II}_3$	(0,0,1)	(0,1,0)	(1,0,0)
$\mathbf{II}_4$	(1,0,0)	(0,0,1)	(0,1,0)
$\mathbf{II}_5$	(0,0,1)	(1,0,0)	(0,1,0)
$\mathbf{II}_6$	(0,1,0)	(0,0,1)	(1,0,0)

The choices of the cases $(\mathbf{I}_i, \mathbf{II}_j)$, $\left(i, j = \overline{1.6}\right)$, give us 36 operators. Such operators are listed as follows:

$$V_1: \begin{cases} x' = ax^2 + ay^2 + az^2 + 2xy \\ y' = bx^2 + by^2 + bz^2 + 2xz \\ z' = 2yz \end{cases} \qquad V_2: \begin{cases} x' = ax^2 + ay^2 + az^2 + 2xy \\ y' = bx^2 + by^2 + bz^2 + 2yz \\ z' = 2xz \end{cases}$$

$$V_3: \begin{cases} x' = ax^2 + ay^2 + az^2 + 2yz \\ y' = bx^2 + by^2 + bz^2 + 2xy \\ z' = 2xz \end{cases} \qquad V_4: \begin{cases} x' = ax^2 + ay^2 + az^2 + 2yz \\ y' = bx^2 + by^2 + bz^2 + 2xz \\ z' = 2xy \end{cases}$$

$$V_5: \begin{cases} x' = ax^2 + ay^2 + az^2 + 2xz \\ y' = bx^2 + by^2 + bz^2 + 2xy \\ z' = 2yz \end{cases} \qquad V_6: \begin{cases} x' = ax^2 + ay^2 + az^2 + 2xz \\ y' = bx^2 + by^2 + bz^2 + 2yz \\ z' = 2xy \end{cases}$$

$$V_7: \begin{cases} x' = bx^2 + by^2 + bz^2 + 2xy \\ y' = ax^2 + ay^2 + az^2 + 2xz \\ z' = 2yz \end{cases} \qquad V_8: \begin{cases} x' = bx^2 + by^2 + bz^2 + 2xy \\ y' = ax^2 + ay^2 + az^2 + 2yz \\ z' = 2xz \end{cases}$$

$$V_9: \begin{cases} x' = bx^2 + by^2 + bz^2 + 2yz \\ y' = ax^2 + ay^2 + az^2 + 2xy \\ z' = 2xz \end{cases} \qquad V_{10}: \begin{cases} x' = bx^2 + by^2 + bz^2 + 2yz \\ y' = ax^2 + ay^2 + az^2 + 2xz \\ z' = 2xy \end{cases}$$

$$V_{11}: \begin{cases} x' = bx^2 + by^2 + bz^2 + 2xz \\ y' = ax^2 + ay^2 + az^2 + 2xy \\ z' = 2yz \end{cases} \qquad V_{12}: \begin{cases} x' = bx^2 + by^2 + bz^2 + 2xz \\ y' = ax^2 + ay^2 + az^2 + 2yz \\ z' = 2xy \end{cases}$$

$$V_{13}: \begin{cases} x' = 2xy \\ y' = ax^2 + ay^2 + az^2 + 2xz \\ z' = bx^2 + by^2 + bz^2 + 2yz \end{cases} \qquad V_{14}: \begin{cases} x' = 2xy \\ y' = ax^2 + ay^2 + az^2 + 2yz \\ z' = bx^2 + by^2 + bz^2 + 2xz \end{cases}$$

$$V_{15}: \begin{cases} x' = 2yz \\ y' = ax^2 + ay^2 + az^2 + 2xy \\ z' = bx^2 + by^2 + bz^2 + 2xz \end{cases} \qquad V_{16}: \begin{cases} x' = 2yz \\ y' = ax^2 + ay^2 + az^2 + 2xz \\ z' = bx^2 + by^2 + bz^2 + 2xy \end{cases}$$

$$V_{17}: \begin{cases} x' = 2xz \\ y' = ax^2 + ay^2 + az^2 + 2xy \\ z' = bx^2 + by^2 + bz^2 + 2yz \end{cases} \qquad V_{18}: \begin{cases} x' = 2xz \\ y' = ax^2 + ay^2 + az^2 + 2yz \\ z' = bx^2 + by^2 + bz^2 + 2xy \end{cases}$$

$$V_{19}: \begin{cases} x' = 2xy \\ y' = bx^2 + by^2 + bz^2 + 2xz \\ z' = ax^2 + ay^2 + az^2 + 2yz \end{cases} \qquad V_{20}: \begin{cases} x' = 2xy \\ y' = bx^2 + by^2 + bz^2 + 2yz \\ z' = ax^2 + ay^2 + az^2 + 2xz \end{cases}$$

$$V_{21}: \begin{cases} x' = 2yz \\ y' = bx^2 + by^2 + bz^2 + 2xy \\ z' = ax^2 + ay^2 + az^2 + 2xz \end{cases} \qquad V_{22}: \begin{cases} x' = 2yz \\ y' = bx^2 + by^2 + bz^2 + 2xz \\ z' = ax^2 + ay^2 + az^2 + 2xy \end{cases}$$

$$V_{23}: \begin{cases} x' = 2xz \\ y' = bx^2 + by^2 + bz^2 + 2xy \\ z' = ax^2 + ay^2 + az^2 + 2yz \end{cases} \qquad V_{24}: \begin{cases} x' = 2xz \\ y' = bx^2 + by^2 + bz^2 + 2yz \\ z' = ax^2 + ay^2 + az^2 + 2xy \end{cases}$$

$$V_{25}: \begin{cases} x' = ax^2 + ay^2 + az^2 + 2xy \\ y' = 2xz \\ z' = bx^2 + by^2 + bz^2 + 2yz \end{cases} \qquad V_{26}: \begin{cases} x' = ax^2 + ay^2 + az^2 + 2xy \\ y' = 2yz \\ z' = bx^2 + by^2 + bz^2 + 2xz \end{cases}$$

$$V_{27}:\begin{cases}x' = ax^2+ay^2+az^2+2yz\\ y' = 2xy\\ z' = bx^2+by^2+bz^2+2xz\end{cases}\quad V_{28}:\begin{cases}x' = ax^2+ay^2+az^2+2yz\\ y' = 2xz\\ z' = bx^2+by^2+bz^2+2xy\end{cases}$$

$$V_{29}:\begin{cases}x' = ax^2+ay^2+az^2+2xz\\ y' = 2xy\\ z' = bx^2+by^2+bz^2+2yz\end{cases}\quad V_{30}:\begin{cases}x' = ax^2+ay^2+az^2+2xz\\ y' = 2yz\\ z' = bx^2+by^2+bz^2+2xy\end{cases}$$

$$V_{31}:\begin{cases}x' = bx^2+by^2+bz^2+2xy\\ y' = 2xz\\ z' = ax^2+ay^2+az^2+2yz\end{cases}\quad V_{32}:\begin{cases}x' = bx^2+by^2+bz^2+2xy\\ y' = 2yz\\ z' = ax^2+ay^2+az^2+2xz\end{cases}$$

$$V_{33}:\begin{cases}x' = bx^2+by^2+bz^2+2yz\\ y' = 2xy\\ z' = ax^2+ay^2+az^2+2xz\end{cases}\quad V_{34}:\begin{cases}x' = bx^2+by^2+bz^2+2yz\\ y' = 2xz\\ z' = ax^2+ay^2+az^2+2xy\end{cases}$$

$$V_{35}:\begin{cases}x' = bx^2+by^2+bz^2+2xz\\ y' = 2xy\\ z' = ax^2+ay^2+az^2+2yz\end{cases}\quad V_{36}:\begin{cases}x' = bx^2+by^2+bz^2+2xz\\ y' = 2yz\\ z' = ax^2+ay^2+az^2+2xy\end{cases}$$

CLASSIFICATION OF THE OPERATORS

In the previous section, we derived 36 QSO which are too many to be investigated one by one. Therefore, we need to classify them into smaller classes so that we can only investigate the elements inside the classes.

Theorem 2. *Let* $\{V_1, \ldots, V_{36}\}$ *be* $\xi^{(a)}$*-QSO given in the previous section. Then they are divided into two non isomorphic classes:*

$L_1 = \{C_1, C_3, C_6, C_7, C_9, C_{12}\}$;

$L_2 = \{C_2, C_4, C_5, C_8, C_{10}, C_{11}\}$;

where

$$C_1=\{V_1,V_{13},V_{31}\},\ C_2=\{V_2,V_{16},V_{35}\},\ C_3=\{V_3,V_{15},V_{33}\},$$
$$C_4=\{V_4,V_{17},V_{32}\},\ C_5=\{V_5,V_{14},V_{34}\},\ C_6=\{V_6,V_{18},V_{36}\},$$
$$C_7=\{V_7,V_{19},V_{25}\},\ C_8=\{V_8,V_{22},V_{29}\},\ C_9=\{V_9,V_{21},V_{27}\},$$
$$C_{10}=\{V_{10},V_{23},V_{26}\},\ C_{11}=\{V_{11},V_{20},V_{28}\},\ C_{12}=\{V_{12},V_{24},V_{30}\}.$$

Proof. Let us first classify the given operators with respect to the renumeration of their coordinates. This means we have to perform $\pi_1^{-1}V\pi_1$ transformation on all the operators. Here

$$\pi_1=\begin{pmatrix}x & y & z\\ y & z & x\end{pmatrix}.$$

We start with V_1 as the first operator, then we get

$$V_1(\pi_1(x,y,z))=V_1(y,z,x)=\begin{pmatrix}ay^2+az^2+ax^2+2yz,by^2\\ +bz^2+bx^2+2xy,2xz\end{pmatrix}$$

So, one can write the last one

$$\pi_1^{-1}V_1\pi_1=\begin{pmatrix}2xy,ax^2+ay^2+az^2+\\ 2xz,bx^2+by^2+bz^2+2yz\end{pmatrix},$$

this means

$$\pi_1^{-1}V_1\pi_1=V_{13}.$$

Now, let us try to do the same for V_{13}, i.e.

$$V_{13}(\pi_1(x,y,z))=V_{13}(y,z,x)=\begin{pmatrix}2yz,ax^2+ay^2+az^2+\\ 2xy,bx^2+by^2+bz^2+2xz\end{pmatrix}.$$

This means

$$\pi_1^{-1}V_{13}\pi_1=\left(bx^2+by^2+bz^2+2xy,2xz,ay^2+az^2+2yz\right).$$

Hence $\pi_1^{-1}V_{13}\pi_1=V_{31}$. Note that one can see that $\pi_1^{-1}V_{31}\pi_1=V_1$. Therefore, we can conclude that V_1, V_{13}, V_{31} are in the same class and we denote it as C_1, i.e. $C_1 = \{V_1, V_{1}3, V_{31}\}$.

By means of the same argument, we conclude that,

$$C_2=\{V_2,V_{16},V_{35}\},\quad C_3=\{V_3,V_{15},V_{33}\},$$
$$C_4=\{V_4,V_{17},V_{32}\},\quad C_5=\{V_5,V_{14},V_{34}\},$$
$$C_6=\{V_6,V_{18},V_{36}\},\quad C_7=\{V_7,V_{19},V_{25}\},$$
$$C_8=\{V_8,V_{22},V_{29}\},\quad C_9=\{V_9,V_{21},V_{27}\},$$
$$C_{10}=\{V_{10},V_{23},V_{26}\},\ C_{11}=\{V_{11},V_{20},V_{28}\},$$
$$C_{12}=\{V_{12},V_{24},V_{30}\}.$$

Now consider another permutations,

$$\pi_2=\begin{pmatrix}x & y & z\\ x & z & y\end{pmatrix},\pi_3=\begin{pmatrix}x & y & z\\ z & y & x\end{pmatrix},\pi_4=\begin{pmatrix}x & y & z\\ y & x & z\end{pmatrix},$$

respectively and perform transformation $\pi_k^{-1}V\pi_k, k=2,3,4$.

Let us take V_1. Then one can see that $\pi_2^{-1}V_1\pi_2=V_{30}$, which yields $C_1\simeq C_{12}$. Similarly, we have $\pi_3^{-1}V_1\pi_3=V_{19}$, $\pi_4^{-1}V_1\pi_4=V_9$, and $\pi_3^{-1}V_1\pi_3=V_{19}$. This means that $C_1\simeq C_7, C_1\simeq C_9$. Moreover, one gets $\pi_3^{-1}V_{30}\pi_3=\pi_4^{-1}V_{19}\pi_4=\pi_2^{-1}V_9\pi_2=V_{33}$, $\pi_4^{-1}V_{30}\pi_4$ $=\pi_2^{-1}V_{19}\pi_2=\pi_3^{-1}V_9\pi_3=V_{18}$, which imply $C_{12}\simeq C_3, C_7\simeq C_3$, and $C_9\simeq C_3$. So, $\{C_1, C_3, C_6, C_7, C_9, C_{12}\}$ are conjugate classes. Let us denote this class by L_1. Using the same argument one can conclude that $L_2 = \{C_2, C_4, C_5, C_8, C_{10}, C_{11}\}$. This completes the proof.

DYNAMICS OF OPERATORS

In this section, for sake of simplicity, we consider the parameters $a = 1, b = 0$.

Class L_1. In this subsection, we are going to study the dynamics of operators taken from L_1. Due to Theorem 2, it is enough to study only one representative taken from the said class. So, we choose V_1, i.e.

$$V_1:\begin{cases} x' = x^2 + y^2 + 2xy \\ y' = 2xz \\ z' = 2yz \end{cases}. \tag{5}$$

To investigate the trajectory or the behavior of this operator, the first step is to find fixed points and the spectrum of Jacobian at these fixed points.
Let us denote

$$\ell_1 = \{(x,y,z) \in S^2 | x = 0\},$$
$$\ell_2 = \{(x,y,z)\} \in S^2 | y = 0\},$$
$$\ell_3 = \{(x,y,z)\} \in S^2 | z = 0\}.$$

Proposition 3. *Let V_1 be given by (5). Then the following statements hold true:*
(i) *$Fix(V_l) = \{(1, 0, 0)\}$. Moreover, the eigenvalues of the Jacobian of V_1 at $(1, 0, 0)$ are $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 2$;*
(ii) *one has $V_1(\ell_1) \subset \ell_2, V_1(\ell_2) \subset \ell_3, V_1(\ell_3) = (1,0,0)$.*

Proof. The fixed point for V_1 is a solution of the equation

$$\begin{cases} x^2 + y^2 + z^2 + 2xy = x \\ 2xz = y \\ 2yz = z \end{cases}.$$

The last ones can be rewritten as follows

$$\begin{cases} x = 2z^2 - 2z + 1, \\ y = 2z(2z^2 - 2z + 1), \\ z = 0, y = 1/2. \end{cases}$$

Hence, if $z = 0$, then $y = 0$, $x = 1$. If $y = 1/2$, then we immediately can solve the last system has no solution belonging to S^2, therefore, one has $Fix(V_1) = \{(1, 0, 0)\}$. We can find that eigenvalues of the Jacobian of V_1 at $(1, 0, 0)$ are $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 2$. This means that the fixed point is hyperbolic.

Let $(x, y, z) \in \ell_1$, i.e. $x = 0$. Then we have $V_1(0, y, z) = (y^2 + z^2, 0, 2yz)$, which means $V_1(0, y, z) \in \ell_2$.

Let $y = 0$, then we have $V_1(x, 0, z) = (x^2 + y^2, 2xz, 0)$, which means $V_1(0, y, z) \in \ell_3$.

Finally, if one lets $z = 0$, then $V_1(x, y, 0) = (x^2 + y^2 + 2xy, 0, 0) = (1, 0, 0)$.

Moreover, one can also check that

$$V_1(1, 0, 0) = V_1(0, 1, 0) = V_1(0, 0, 1) = (1, 0, 0).$$

We know that $z = 1 - x - y$ and $x + y = 1 - z$. So

$$x' = x^2 + y^2 + z^2 + 2xy = (1 - z)^2 + z^2$$
$$= 2(z - 1/2)^2 + 1/2. \tag{6}$$

This means $x' \geq 1/2$ for any $(x, y, z) \in S^2$. Therefore, we divide the region $x \geq \frac{1}{2}$ into two S_1 and S_2 subregions. Namely,

$$S_1 = \left\{(x,y,z) \in S^2 \middle| x > y > z, x \geq \frac{1}{2}\right\},$$
$$S_2 = \left\{(x,y,z) \in S^2 \middle| x > z > y, x \geq \frac{1}{2}\right\}.$$

Theorem 4. *Let V_1 be given by (5). Then the following statements hold true:*
(i) one has $V_1(S_1) \subset S_1$;
(ii) one has $V_1(S_2)$ S_1;
(iii) The functional $\varphi(x, y, z) = x$ is a Lyapunov function on $x \geq \frac{1}{2}$;
(iv) *For any $(x, y, z) \in S^2$ one has* $\lim_{n\to\infty} V_1^n(x,y,z) = (1,0,0)$.

Proof.

(i) Let $(x, y, z) \in S_1$, i.e. $x > y > z$, $x \geq 1/2$. This implies that y' $2xz > 2yz = z'$. Therefore, $y' > z'$. It is clear that $2xy > 2xz$, which yields

$$x' = x^2 + y^2 + z^2 + 2xy > 2xz = y'.$$

This means $x' > y'$, so keeping in mind (6), one finds $V_1(S_1) \subset S_1$.

(ii) Let $(x, y, z) \in S_2$, i.e. $x > z \geq y$, $x \geq 1/2$. This means that $y' = 2xz \geq 2yz = z'$. From (6) and $x' + y' + z' = 1$ one gets

$$1 - (y' + z') \geq \frac{1}{2},$$

which means $y' + z' \leq \frac{1}{2}$. Therefore, we have

$$y' \leq y' + z' \leq \frac{1}{2} \leq x',$$

so, $y' < x'$. Hence, one gets $V_1(S_2) \subset S_1$.

(iii) Let us consider the following functional

$$\tilde{\varphi}(x,y,z) = y + z.$$

We want to show that

$$\tilde{\varphi}(x',y',z') \leq \tilde{\varphi}(x,y,z), \tag{7}$$

which means $y' + z' \leq y + z$, i.e. $2xz + 2yz \leq y + z$, the last one is true due to

$$2z(x + y) \leq 2z(x + y + z) = 2z = z + z \leq y + z.$$

From $x = 1 - y - z$ we find $\varphi(x, y, z) = 1 - \tilde{\varphi}(x,y,z)$. Therefore, (7) implies that $\varphi(V_1(x)) \leq \varphi(x)$.

Hence, $\{\varphi(V_1^n(x))\}$ is a decreasing and bounded sequence. So, it converges, i.e. $\varphi(V_1^n(x)) \to C$ for any $(x, y, z) \in S^2$ with $x \geq 1/2$.

(iv) Let us show that the functional $\varphi_1(x, y, z) = z$ is also a Lyapunov one over $x \geq 1/2$. Indeed, from $y \leq \frac{1}{2}$, one has $\varphi_1(x', y', z') = z' = 2yz \leq z = \varphi_1(x, y, z)$ which is the desired assertion. This with (iii) implies that the sequences $x^{(n)}$ and $z(n)$ are convergent, therefore, the sequence $\{y^{(n)}\}$ is also convergent. Assume that $(x^n, y^n, z^n) \to (C_1, C_2, C_3)$. One can see that (C_1, C_2, C_3) is a fixed point. So, due to Proposition 3 we conclude that $V_1^{\,n}(x, y, z)$ converges to $(1, 0, 0)$.

Class L_2. In this subsection, we are going to study the dynamics of operators taken from the class L_2. Let us investigate the trajectory of the operator V_5 given by

$$V_5\begin{cases} x' = x^2 + y^2 + z^2 + 2xz \\ y' = 2xy \\ z' = 2yz \end{cases}. \tag{8}$$

Proposition 5. *Let V_5 be given by (8). Then the following statements hold true:*

(i) $Fix(V_5) = \{1, 0, 0), (1/2, 1/2, 0)\}$.

(ii) *one has* $V_5(\ell_1) \subset \ell_2$, $V_5(\ell_2) = (1, 0, 0)$, $V_5(\ell_3) = \ell_3$.

Proof.

(i) In order to find fixed point of (8) we shall solve the following system of equations:

$$\begin{cases} x = x^2 + y^2 + z^2 + 2xz \\ y = 2xy \\ z = 2yz \end{cases}. \tag{9}$$

Now, consider second equation from system (9). If $y = 0$ it is clear that $z = 0$. Therefore, $x = 1$. If $y \neq 0$, then $x = \frac{1}{2}, z = 0$ and $y = \frac{1}{2}$. Hence, $Fix(V_5) = \{(1, 0, 0), (\frac{1}{2}, \frac{1}{2}, 0)\}$.

(ii) First of all let $(x, y, z) \in \ell_1$ be an initial point, i.e. $x = 0$, then $V_5(0, y, z) = \{y^2 + z^2, 0, 2yz\}$ the image lies in yz–plane. Therefore, $V_5(\ell_1) \subset \ell_2$. Next let $(x, y, z) \in \ell_2$ be an initial point, i.e. $y = 0$, then $V_5(x, 0, z) = \{x^2 + z^2 + 2xz, 0, 0\} = (1, 0, 0)$. Finally, we let $(x, y, z) \in \ell_3$ be an initial point, i.e. $z = 0$, then $V_5(x, y, 0) = \{x^2 + y^2, 2xy, 0\}$ the image lies in xy–plane. Therefore, $V_5(\ell_3) \subset \ell_3$. Moreover, $V_5(0, 1, 0) = V_5(0, 0, 1) = V_5(1, 0, 0) = (1, 0, 0)$.

Let us denote

$$\operatorname{int} S^2 = \{(x,y,z) \in S^2 \,|\, xyz > 0\},$$
$$\partial S^2 = \{(x,y,z) \in S^2 \,|\, xyz = 0\}$$

Theorem 6. *Let V_5 be given by (8). The following assertions hold true:*

(i) *for any* $(x, y, z) \in \partial S^2$ *one has* $\lim_{n\to\infty} V_5^n(x,y,z) = (1,0,0)$;

(ii) *for any* $(x, y, z) \in intS^2$, *one has* $\lim_{n\to\infty} V_5^n(x,y,z) = (\frac{1}{2},\frac{1}{2},0)$.

Proof.

(i) In fact, by using Proposition 5 we have that the trajectory for any point taken from ℓ_1 goes to ℓ_2 and the trajectory for any point taken from ℓ_2 goes to (1, 0, 0). So, it is enough for us to study the dynamic of (8) over the line ℓ_3. To do that we have to consider the second coordinate of (8). Namely, $y = f(y) = 2y(1 - y)$. Now, let us divide ℓ_3 into to sets as the follows: $I_1 = [0,\frac{1}{2}]$ and $I_2 = (\frac{1}{2},1]$. One can find that $f(y)$ increasing over I_1 and decreasing over I_2. Hence, one gets that $f(I_1) \subseteq I_1$, and $f(I_2) \subseteq I_1$. So, it is enough to study the dynamic of $f(y)$ over I_1. In order to do that we consider the function $f(y) - y = 2y(1 - y) - y$. One can check that $f(y) - y \le 0$, $\forall y \in I_1$, since $f(y)$ increasing over I_1. Therefore, one finds $f^{\,n+1}(y) \le f^{\,n}(y)$. So, $\{(f^{\,n}(y)\}$ is decreasing and bounded. Moreover, $\{f^{\,n}(y)\}$ converges to y, which is a fixed point of $f(y)$, and only possibility is $y = 0$. Hence, x^n converges to 1. Thus, we get the desired assertion.

(ii) Let us define the function,

$\varphi(x, y, z) = y + z$

Then we have

$\varphi(x'. y', z') = y' + z' = 2y(x + z)$

$= 2y(1 - y)$

One can find that the maximum value for the function $\phi = (x', y', z') = 2y(1 - y)$ in the interval [0,1] is $\frac{1}{2}$ Therefore, we have the following

(1) If $1 \ge y \ge \frac{1}{2}$, it implies $y' + z' \le \frac{1}{2}$ which gives $y' \le \frac{1}{2}$.

(2) If $0 \le y \le \frac{1}{2}$, it implies that $y' + z' \le \frac{1}{2}$ which also gives $y' \le \frac{1}{2}$.

From these statements, by defining

$$\tilde{D} = \left\{x \in S^2 \,\middle|\, y + z\frac{1}{2}\right\},$$

one gets $V_5(\tilde{D}) \subset \tilde{D}$. This means $\tilde{D}$ is an invariant set, and for set

$$D = \left\{x \in S^2 \,\middle|\, y + z \ge \frac{1}{2}\right\},$$

we have $V_5(D) \subset \tilde{D}$. So, it is enough to study the dynamics over the set $\tilde{D}$.

We know that

$$x' = x^2 + y^2 + z^2 + 2xz. \; x \ge \frac{1}{2},$$

then $y' = 2xy \ge y$. Therefore, one finds $y^{(n+1)} \ge y^{(n)}$ for every $n \in \mathbb{N}$. Hence, the limit $\lim_{n\to\infty} y^{(n)} = y^*$ exists.

Next, we want to show that $z^{(n)}$ converges. Indeed, due to $y \le \frac{1}{2}$ one gets $z' = 2yz \le z$. This implies $z' \le z$, therefore, $z^{(n+1)} \le z^{(n)}$. So, the limit $\lim_{n\to\infty} z^{(n)} - z^{(n)}$ exists.

We have showed that both $y^{(n)}$ and $z^{(n)}$ converge, therefore $x^{(n)} = 1 - y^{(n)} - z^{(n)}$ also converges. We define the function $\varphi(x, y, z) = xy$.

One can see that $2x \ge 1$ implies $x' \ge \frac{1}{2}$. Therefore, we immediately get $x'y' \ge xy$. Hence,

$$\varphi(V_5(x, y, z)) \ge \varphi(x, y, z).$$

So, $x^{(n)} y^{(n)} \geq x^{(n-1)} y^{(n-1)}$, i.e. $x^{(n)} y^{(n)}$ is an increasing sequence, therefore, $x^{(n)} y^{(n)}$ must not be convergent to 0. So, the limit should be a positive number. Therefore, $\lim_{N\to\infty} x^{(n)} y^{(n)} = x^{(*)} y^{(*)}$, therefore $x^* = y^* \neq 0$. Hence we have the only fixed point which is $x^* = y^* = \frac{1}{2} \Rightarrow (x^*, y^*, z^*) = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$.

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Department of Computational & Theoretical Sciences
Faculty of Science, International Islamic University Malaysia
P.O. Box, 141, 25710, Kuantan
Pahang, Malaysia

*Corresponding author; email: farrukh_m@iium.edu.my

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