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On P-Convergence of Four Dimensional Weighted Sums of Double Random Variables

(Hasil Tambah Berpemberat Empat Dimensi Berganda Pemboleh Ubah Rawak ke atas Penumpuan-P)

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ABSTRACT

The goal of this paper was to present a series of limit theorems that characterizes independent double random variables via four dimensional summability transformation. In order to accomplish this goal we began with the presentation of the following theorem that characterize pairwise independent random variables: let $[x_{k,l}]$ *be a double sequence of pairwise independent random variables such that* $[x_{k,l}]$ *was uniformly integrable. Let* $[a_{m,n,k,l}]$ *be a four dimensional matrix such that*

$\sum_{k,l=1,1}^{\infty,\infty}|a_{m,n,k,l} \leq C$ *for all ordered pair* (m, n) *and for some C and*

$\max_{k,l}|a_{m,n,k,l}|$ *converges to 0 in probability*

Then $\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}(x_{k,l} - E(x_{k,l}))$ *converges in mean to* 0. *Other extensions and variations via multidimensional transformation shall also be presented.*

Keywords. Double sequences Pringsheim limit point; P-convergent; RH-Regular

ABSTRAK

Penyelidikan ini bertujuan untuk membentangkan satu siri teorem had yang mencirikan pemboleh ubah rawak bebas berganda melalui keterhasiltambahan transformasi empat dimensi. Untuk mencapai matlamat ini, kami mulakan dengan memberikan teorem yang mencirikan pasangan demi pasangan pemboleh ubah rawak: biar $[x_{k,l}]$ *menjadi jujukan ganda dua pasangan demi pasangan pemboleh ubah rawak bebas supaya* $[x_{k,l}]$ *menjadi seragam terkamir. Biar* $[a_{m,n,k,l}]$ *menjadi empat dimensi matriks supaya*

$\sum_{k,l=1,1}^{\infty,\infty}|a_{m,n,k,l} \leq C$ *untuk semua pasangan yang disusun* (m, n) *dan bagi sesetengah C dan*

$\max_{k,l}|a_{m,n,k,l}|$ *penumpuan dalam kebarangkalian kepada 0*

Kemudian $\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}(x_{k,l} - E(x_{k,l}))$ *menumpu pada min untuk* 0. *Perluasan lain dan variasi melalui transformasi bermultimatra turut dikemukakan.*

Kata kunci: Jujukan ganda dua titik had Pringsheim; penumpuan P; RH biasa

INTRODUCTION

In 1966 Pruitt has presented limit theorems examining independent random variables via summability theory. Patterson and Savas (2008) extended Pruitt's results to double sequences using four-dimensional transformations. The goals of this paper included the continuation of Pruitt's analysis by examine double sequences $[x_{k,l}]$ of random variables defined on a probability space (Ω, β, P) and four-dimensional matrix $\{a_{m,n,k,l} : 1,2,3,\ldots\}$ of real numbers using the following transformations,

$$\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} x_{k,l}.$$

Throughout this paper all limits will be taking the Pringsheim sense. In order to accomplish the goals set forth in this paper, we began with the following two theorems: First, let $[x_{k,l}]$ be a double sequence of pairwise independent random variables such that $[x_{k,l}]$ is uniformly integrable. Let $[a_{m,n,k,l}]$ be a four dimensional matrix such that

$\sum_{k,l=1,1}^{\infty,\infty}|a_{m,n,k,l}|^r \leq C$ for all ordered pair (m, n) and for some C; and $\max_{k,l}|a_{m,n,k,l}|$ converges to 0 in probability

Then $\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}(x_{k,l} - E(x_{k,l}))$ converges in mean to 0. While this theorem cannot be extended to separable Banach spaces, the following can be extended to such spaces.

Second, let $[x_{k,l}]$ be a double sequence of real random variables defined on a probability space (Ω, β, P) such that $|x_{k,l}|^r$; $k, l = 1, 2, 3, \ldots$ in uniformly integrable for some $0<r<1$. Let $\{a_{m,n,k,l} : = 1, 2, 3, \ldots\}$ be a four dimensional matrix of real numbers satisfying the following properties:

$\sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}|^r \leq C$ for every ordered pair (m, n) and C some positive constant; and

$\max_{k,l} |a_{m,n,k,l}|$ is converges to zero

Then $\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} x_{k,l}$ converges to 0 in r^{th} mean.

Definitions, Notations, and Preliminary Results

Before continuing with this paper we present some definitions and preliminaries.

Definition 2.1 (Pringsheim 1900). A double sequence $x = [x_{k,l}]$ has Pringsheim limit L (denoted by $\lim x = L$) provided that given $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > N$.

In order to present the notion of divergent double sequence one should observe the following notion of subsequence of a double sequence.

Definition 2.2 (Patterson 2000). The double sequence y is a double subsequence of x provided that there exist increasing index sequences $\{n_j\}$ and $\{k_j\}$ such that if $x_j = x_{n_j, k_j}$, then y is formed by

$$\begin{array}{cccc} x_1 & x_2 & x_5 & x_{10} \\ x_4 & x_3 & x_6 & - \\ x_9 & x_8 & x_7 & - \\ - & - & - & -. \end{array}$$

A number β is called a Pringsheim limit point of the double sequence $x = [x_{n,k}]$ provided that there exists a subsequence $y = [y_{n,k}]$ of $[x_{n,k}]$ that has Pringsheim limit $\beta : \lim y_{n,k} = \beta$. With this definition we can finally presented a complementary notions of convergence double, that was, a double sequence x is divergent in the Pringsheim sense provided that x does not converge in the Pringsheim sense. For more recent developments on double sequences one can consult the papers (Patterson 2000; Patterson & Savas 2013, 2012a, 2012b, 2012c, 2011, 2010a, 2010b, 2008) where more references can be found.

Results

Theorem 3.1. Let $[x_{k,l}]$ be a double sequence of pairwise independent random variables such that $[x_{k,l}]$ is uniformly integrable and let $[a_{m,n,k,l}]$ be a four dimensional matrix such that $\sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}| \leq c$ for all ordered pair (m, n) and for some c and $\max_{k,l} |a_{m,n,k,l}|$ converges to 0 in probability, then $\sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l} (x_{k,l} - E(x_{k,l}))$ converges in mean to 0.

Proof. It is clear that $\sum_{k,l=1,1} a_{m,n,k,l} \left(x_{k,l} - E\left(x_{k,l}\right)\right)$ converges absolutely a.e. in probability for every ordered pair (m, n), since $\sup_{k,l} E|x_{k,l}| < \infty$ and $\sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}$ converges. Let $t>0$. We will show that,

$$\lim_{m,n} \mathrm{P}\left\{\left|\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}(x_{k,l} - E(x_{k,l}))\right| > t\right\} = 0$$

Let $\epsilon > 0$. Since $\{x_{k,l} : k, l = 1, 2, 3, \ldots\}$ is uniformly integrable there exists $\delta > 0$ such that,

$$\sup_{k,l} \int_A |x_{k,l}| dP < \frac{\epsilon t}{8c} \qquad (3.1)$$

where $A \in \beta$ and $P(A) < \delta$. By the Chebychev's Inequality for any $m > 0$ and $k, l = 1, 2, 3, \ldots$

$$P\left\{|x_{k,l}| > m\right\} \leq \frac{1}{m} E|x_{k,l}|$$
$$\leq \frac{1}{m} \sup_{k,l} E|x_{k,l}|.$$

Consequently, there exists $a > 0$ such that

$$sup_{k,l}\, \mathrm{P}(|x_{k,l}| > a) < \delta. \qquad (3.2)$$

Define for each (k, l) the following

$$y_{k,l} = \begin{cases} x_{k,l}, & \text{if } |x_{k,l}| \leq a; \\ 0, & \text{if } otherwise \end{cases}$$

and $z_{k,l} = x_{k,l} - y_{k,l}$. Note that $[y_{k,l}]$ is a double sequence of pairwise independent random variables satisfying

$$|y_{k,l} - E(y_{k,l})| \leq 2a$$

for every $k, l = 1, 2, 3, \ldots$. By 3.1 and 3.2 we have for every $k, l = 1, 2, 3, \ldots$

$$E|z_{k,l}| = \int_{\{|x_{k,l} \geq a|\}} |x_{k,l}| dP < \frac{\epsilon t}{8c}.$$

Thus for every ordered pair (m, n)

$$E\left|\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}(z_{k,l} - E(z_{k,l}))\right| \leq \sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}| E|z_{k,l} - E(z_{k,l})|$$
$$\leq 2\sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}| E|z_{k,l}|$$
$$\leq \frac{\epsilon t}{4}.$$

By the Chebychev's Inequality for every (m, n)

$$P\left\{\left|\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}(z_{k,l} - E(z_{k,l}))\right| > \frac{t}{2}\right\} < \frac{\epsilon}{2}. \qquad (3.3)$$

Next, we choose $\bar{M}$ and $\bar{N}$ such that for every $m > \bar{M}$ and $n > \bar{N}$ we have

$$\max_{k,l}\left|a_{m,n,k,l}\right| < \frac{\eth t^2}{32a^2c}.$$

By the linearity properties of variance and the covariance property of independent random variables we are granted the following for $m, n > \bar{M}, \bar{N}$.

$$P\left\{\left|\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}(y_{k,l} - E(y_{k,l}))\right| > \frac{t}{2}\right\} \le$$
$$\frac{4}{t^2}\operatorname{Var}\left(\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}(y_{k,l} - E(y_{k,l}))\right) \le$$
$$\frac{4}{t^2}\sum_{k,l=1,1}^{\infty,\infty} a^2_{m,n,k,l}E(y_{k,l} - E(y_{k,l}))^2 =$$
$$\frac{4}{t^2}\max_{k,l}\left|a_{m,n,k,l}\right|\sum_{k,l=1,1}^{\infty,\infty}\left|a_{m,n,k,l}\right|E(y_{k,l} - E(y_{k,l}))^2 \le \frac{\delta}{2}. \tag{3.4}$$

Finally, 3.3 and 3.4 yield the following

$$P\left\{\left|\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}(x_{k,l} - E(y_{k,l}))\right| > t\right\} \le$$
$$P\left\{\left|\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}(y_{k,l} - E(y_{k,l}))\right| > \frac{t}{2}\right\} +$$
$$P\left\{\left|\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}(z_{k,l} - E(y_{k,l}))\right| > \frac{t}{2}\right\} < \varepsilon$$

for every $m, n > \bar{M}, \bar{N}$. Thus \$$\sum_{k,l=1,1}^{\infty,\infty}\left|a_{m,n,k,l}(x_{k,l} - E(x_{k,l}))\right.$ for $m \ge 1, n \ge 1$ converges to 0 in probability. Now for converges in mean it is sufficient to show that $\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}(x_{k,l} - E(x_{k,l}))$ is uniformly integrable. Now since $[x_{k,l}]$ is uniformly $\sum_{k,l=1,1}^{\infty,\infty}\left|a_{m,n,k,l}\right|$ integrable and $\sum_{k,l=1,1}^{\infty,\infty}\left|a_{m,n,k,l}\right|$ is bounded it is clear that $\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}(x_{k,l} - E(x_{k,l}))$ is uniformly integrable.

The following theorem is an extension of Theorem 3.1 to r^{th}-mean.

Theorem 3.2. Let $[x_{k,l}]$ be a double sequence of real random variables defined on a probability space (Ω, β, P) such that $\left|x_{k,l}\right|^r; k, l = 1,2,3,\ldots$ in uniformly integrable for some $0 < r < 1$. Let $\{a_{m,n,k,l} : m, n, k, l = 1, 2, 3, \ldots\}$ be a four dimensional matrix of real numbers satisfying the following properties:

$\sum_{k,l=1,1}^{\infty,\infty}\left|a_{m,n,k,l}\right|^r \le C$ for every ordered pair (m, n) and C some positive constant;

$\max_{k,l}\left|a_{m,n,k,l}\right.$ is converges to zero.

Then $\sum_{k,l=1,1}^{\infty,\infty}\left|a_{m,n,k,l}\, x_{k,l}\right.$ converges to 0 in r^{th} mean.

Proof. Observe that $\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}x_{k,l}$ converges absolutely a.e. in probability for every ordered pair (m, n), since $\sum_{k,l=1,1}^{\infty,\infty}\left|a_{m,n,k,l}\right|^r\left|x_{k,l}\right|^r$ converges a.e. in probability for $0 < r < 1$. Let $t > 0$. We will show that

$$\mathrm{P}-\lim_{m,n}\mathrm{P}\left\{\left|\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}x_{k,l} > t\right|\right\} = 0.$$

Let $\eth > 0$. Since $\{x_{k,l} : k, l = 1,2,3,\ldots\}$ is uniformly integrable there exists $\delta > 0$ such that

$$\sup_{k,l}\int_A\left|x_{k,l}\right|^r dP < \frac{\eth t}{4C} \tag{3.5}$$

where $A \in \beta$ and $P(A) < \delta$. Similar to Theorem 3.1 for any $m > 0$ we are granted the following by the Chebychev's Inequality for $k, l = 1, 2, 3, \ldots$

$$\mathrm{P}\left\{\left|x_{k,l}\right| > m\right\} \le \frac{1}{m}E\left|x_{k,l}\right|$$
$$\le \frac{1}{m}\sup_{k,l}E\left|x_{k,l}\right|.$$

Consequently, there exists $a > 0$ such that

$$\sup_{k,l}\mathrm{P}\left(\left|x_{kl}\right| > a\right) < \delta. \tag{3.6}$$

Define for each (k, l) the following

$$y_{k,l} = \begin{cases} x_{k,l}, & \text{if } \left|x_{k,l}\right| \le a; \\ 0, & \text{if otherwise} \end{cases}$$

and $z_{k,l} = x_{k,l} - y_{k,l}$. Note that $[y_{k,l}]$ is a double sequence of pairwise independent random variables satisfying

$$\left|y_{k,l}\right| \le a$$

for every $k, l = 1, 2, 3, \ldots$. By 3.5 and 3.6 we have for every $k, l = 1, 2, 3, \ldots$

$$E\left|z_{k,l}\right|^r = \int_{\{\left|x_{k,l}\right| \ge a\}}\left|x_{k,l}\right|^r dP < \frac{\sqrt{\eth t}}{2c}.$$

Also note that $\left|x_{k,l}\right|^{1-r}; k, l = 1,2,3,\ldots$ in uniformly integrable for some $0 < r < 1$.

Thus

$$E\left|z_{k,l}\right|^{1-r} = \int_{\{\left|x_{k,l}\right| \ge a\}}\left|x_{k,l}\right|^{1-r} dP < \frac{\sqrt{\eth t}}{2c}.$$

Thus for every ordered pair (m, n)

$$E\left|\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} z_{k,l}\right| \le \sum_{k,l=1,1}^{\infty,\infty} \left|a_{m,n,k,l}\right|^{1-r} E\left|x_{k,l}\right|^{1-r} \left|a_{m,n,k,l}\right|^{r} E\left|z_{k,l}\right|^{r}$$

$$\le \sup_{k,l}\left|a_{m,n,k,l}\right|^{1-r} \sum_{k,l=1,1}^{\infty,\infty} E\left|x_{k,l}\right|^{1-r} \left|a_{m,n,k,l}\right|^{r} E\left|z_{k,l}\right|^{r}$$

$$\le \frac{\grave{o}t}{4}.$$

By the Chebychev's Inequality for every (m, n)

$$P\left\{\left|\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} z_{k,l}\right| > \frac{t}{2}\right\} < \frac{\grave{o}}{2}. \tag{3.7}$$

Next, we choose $\bar{M}$ and $\bar{N}$ such that for every $m > \bar{M}$ and $n > \bar{N}$ we have

$$\max_{k,l}\left|a_{m,n,k,l}\right|^{1-r} < \frac{\grave{o}t}{4aC}$$

Now observe that for $m, n > \bar{M}, \bar{N}$ we have the following

$$P\left\{\left|\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} y_{k,l}\right| > \frac{t}{2}\right\} \le \frac{2}{t} E\left|\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} y_{k,l}\right|$$

$$\le \frac{2a}{t}\sum_{k,l=1,1}^{\infty,\infty}\left|a_{m,n,k,l}\right|$$

$$= \frac{2a}{t}\sum_{k,l=1,1}^{\infty,\infty}\left|a_{m,n,k,l}\right|^{1-r}\left|a_{m,n,k,l}\right|^{r}$$

$$\le \frac{2a}{t}\max_{k,l}\left|a_{m,n,k,l}\right|^{1-r}\sum_{k,l=1,1}^{\infty,\infty}\left|a_{m,n,k,l}\right|^{r}$$

$$\le \frac{\grave{o}}{2}. \tag{3.8}$$

Finally, 3.7 and 3.8 yield the following

$$P\left\{\left|\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} x_{k,l}\right| > t\right\} \le P\left\{\left|\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} y_{k,l}\right| > \frac{t}{2}\right\}$$

$$+P\left\{\left|\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} z_{k,l}\right| > \frac{t}{2}\right\}$$

$$< \varepsilon$$

for every $m, n > \bar{M}, \bar{N}$. Thus $\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} x_{k,l}$ for $m > 1$, $n > 1$ converges to 0 in probability. For converges in mean it is sufficient to show that $\left|\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} x_{k,l}\right|^{r}$ is uniformly integrable. Now since $[\left|x_{k,l}\right|^{r}]$ is uniformly integrable and $\sum_{k,l=1,1}^{\infty,\infty}\left|a_{m,n,k,l} x_{k,l}\right|^{r}$ is also bounded it is clear that $\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} x_{k,l}$ is uniformly integrable.

It should be noted that *Theorem 3.2* is also valid for separable Banach spaces similar to Wang and Rao's (1985) result. The general results is stated below.

Theorem 3.3. Let $[x_{k,l}]$ be a double sequence of real random variables defined on separable Banach spaces B with a norm $|\cdot|$ such that $x_{k,l}^{r}; k,l = 1,2,3,\ldots$ in uniformly integrable for some $0 < r < 1$. Let $\{a_{m,n,k,l} : m, n, k, l = 1, 2, 3,\ldots\}$ be a four dimensional matrix of real numbers satisfying the following properties:

$\sum_{k,l=1,1}^{\infty,\infty}\left|a_{m,n,k,l}\right|^{r} \le C$ for every ordered pair (m, n) and C some positive constant;

$\max_{k,l}\left|a_{m,n,k,l}\right|$ is converges to zero.

Then $\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} x_{k,l}$ converges to 0 in r^{th} mean.

However, *Theorem 3.1* was not valid on separable Banach spaces, as usual we cannot estimate the convergence of $\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} y_{k,l}$.

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