Sains Malaysiana 53(6)(2024): 1427-1440
http://doi.org/10.17576/jsm-2024-5306-16

On a New Transmuted Three-Parameter Lindley Distribution and Its Applications

(Taburan Lindley Tiga Parameter Baharu yang Diubahsuai dan Pengaplikasiannya)

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Received: 6 February 2024/Accepted: 7 May 2024

ABSTRACT

In this paper, a new transmuted three-parameter Lindley distribution (TTHPLD) is established using the transmutation map method, which includes the Lindley distribution, two-parameter Lindley distribution, transmuted two-parameter Lindley distribution and three-parameter Lindley distribution as special cases. The statistical properties of the TTHPLD model, which are based on moments, order statistics, hazard rate functions, reliability functions, and Renyi entropy, have been studied. Moreover, the maximum likelihood estimators (MLEs) of the TTHPLD are obtained via differential evolution algorithms, and a simulation study is conducted to evaluate the consistency of the MLEs. Finally, the proposed distribution is applied to a real dataset and compared with other well-known models.

Keywords: Differential evolution algorithm; generalized Lindley distribution; hazard rate function; Renyi entropy; maximum likelihood estimation

ABSTRAK

Dalam kertas ini, sebuah taburan Lindley tiga-parameter terbaharu (TTHPLD) yang ditransmutasikan telah dibangunkan menggunakan kaedah pemetaan transmutasi, yang merangkumi taburan Lindley, taburan Lindley dua-parameter, taburan Lindley dua-parameter tertransmutasi dan taburan Lindley tiga-parameter sebagai kes khas. Sifat-sifat statistik model TTHPLD yang berdasarkan momen, statistik perintah, fungsi kadar bahaya, fungsi kebolehpercayaan, dan entropi Renyi telah dikaji. Selain itu, penganggar kemungkinan maksimum (MLE) bagi TTHPLD diperoleh melalui algoritma evolusi berbeza dan kajian simulasi dijalankan untuk menilai kekonsistenan MLE tersebut. Akhirnya, pengedaran yang dicadangkan ini diaplikasikan kepada set data sebenar dan dibandingkan dengan model terkenal yang lain.

Kata kunci: Algoritma evolusi berbeza; anggaran kebolehjadian maksimum; fungsi kadar bahaya; entropi Renyi; taburan Lindley teritlak

INTRODUCTION

The Lindley distribution (LD) proposed by Lindley (1958) is a useful tool in statistical modeling for data with nonnegative values and decreasing hazard rates. Its applications include survival analysis, reliability engineering, actuarial science, and insurance claim modeling. For example, in reliability engineering, LD is often used to model the failure times of mechanical components or the time to failure. Moreover, in actuarial science and insurance claim modeling, it can also capture the nonnegative nature of claim amounts and the decreasing hazard rates associated with certain types of claims.

If a random variable X follows the Lindley distribution with parameter θ, its probability density function (pdf) and cumulative distribution function (cdf) are defined as

$$f(x;\theta) = \frac{\theta^2}{1+\theta}(1+x)e^{-\theta x} \text{ and } F(x;\theta) = 1 - \left(\frac{1+\theta+\theta x}{1+\theta}\right)e^{-\theta x},$$

where $x > 0$ and $\theta > 0$. The structural properties, estimations, and goodness-of-fit tests for the LD can be found in Ghitany, Atieh and Nadarajah (2008). Furthermore, they showed that the LD is superior to the exponential distribution in terms of waiting time before bank customer service.

In practical analysis, the collected data are often heterogeneous, with various parts of the data following different distributions. Although the LD is excellent at modeling heavy-tailed data features with extreme values or outliers, the distribution needs to be further expanded to accommodate the heterogeneity of the data. Currently, as a well-established distribution expansion technique, the transmutation map method (Shaw & Buckley 2009) is widely used to extend known distributions. These distributions include the Rayleigh distribution and transmuted Rayleigh distribution (Merovci 2013b), the Weibull distribution and transmuted Weibull distribution (Aryal & Tsokos 2011), the Pareto distribution and transmuted Pareto distribution (Merovci & Puka 2014), the Gompertz distribution and transmuted Gompertz distribution (Moniem & Seham 2015), the Lindley distribution and transmuted Lindley distribution (Al-Khazaleh, Al-Omari & Al-Khazaleh 2016; Merovci 2013a), the normal distribution and transmuted normal distribution (Ieren & Abdullahi 2020), and the geometric distribution and transmuted geometric distribution (Chakraborty 2015). In their analysis, the transformed distribution can enhance data fitting and prediction.

Generally, a random variable X is said to have a transmuted distribution if its cdf is given by

$$F(x) = (1+\lambda)G(x) - \lambda[G(x)]^2, |\lambda| \leq 1,$$

with the corresponding pdf defined as

$$f(x) = g(x)[1+\lambda-2\lambda G(x)],$$

where $G(x)$ represents the cdf of the base distribution and pdfs $f(x)$ and $g(x)$ correspond to the cdfs $F(x)$ and $G(x)$, respectively. Hence, the pdf and cdf of the transmuted Lindley distribution (TLD) proposed by Merovci (2013a) are as follows:

$$f(x;\theta,\lambda) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\left(1+\lambda-2\lambda\left(1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x}\right)\right),$$

$$F(x;\theta,\lambda) = \frac{e^{-2\theta x}\left(\left(e^{\theta x}-1\right)\theta+\left(e^{\theta x}-\theta x-1\right)\right)\left(e^{\theta x}(1+\theta)+(1+\theta+\theta x)\lambda\right)}{(1+\theta)^2},$$

where $x > 0$, $\theta > 0$ and $|\lambda| \leq 1$. Moreover, Abouammoh, Alshangiti and Ragab (2015), Ekhosuehi, Opone and Odobaire (2018), Shanker and Rahman (2020) and Shanker and Sharma (2013), proposed various two-parameter Lindley distributions. As a representative distribution with wide application, the pdf and cdf of the two-parameter Lindley distribution (TPLD) proposed by Shanker and Sharma (2013) with shape parameters θ,α are given by

$$f(x;\alpha,\theta) = \frac{\theta^2}{\theta+\alpha}(1+\alpha x)e^{-\theta x} \text{ and } F(x;\alpha,\theta) =$$

$$1-\frac{\theta+\alpha+\alpha\theta x}{\theta+\alpha}e^{-\theta x},$$

where $x > 0$, $\theta > 0$ and $\alpha\,\theta > -1$.

According to real data analysis, the TPLD provides closer fits than does the LD. Hence, Al-Khazaleh, Al-Omari and Al-Khazaleh (2016) derived a new transmuted two-parameter Lindley distribution (TTPLD) based on the TPLD. The pdf and cdf of the TTPLD, with shape parameters θ, α, λ, are given by

$$f(x;\theta,\alpha,\lambda) = \frac{\theta^2}{\theta+\alpha}(1+\alpha x)e^{-\theta x}\left[1+\lambda-2\lambda\left(1-\frac{\theta+\alpha+\alpha\theta x}{\theta+\alpha}e^{-\theta x}\right)\right],$$

$$F(x;\theta,\alpha,\lambda) = \frac{e^{-2\theta x}\left[\left(e^{\theta x}-1\right)\theta+\alpha\left(e^{\theta x}-\theta x-1\right)\right]\left[e^{\theta x}(\alpha+\theta)+(\alpha+\theta+\alpha\theta x)\lambda\right]}{(\alpha+\theta)^2},$$

where $x > 0$, $\theta > 0$ and $\alpha > -\theta$. Moreover, Khazaleh et al. (2016) showed that the TTPLD offers a superior fit to the data compared to the basic LD and TPLD.

In recent years, a new more flexible three-parameter Lindley distribution (THPLD) proposed by Shanker et al. (2017) and its extension (Shanker, Shukla & Mishra 2017) and applications (Al-Omari, Ciavolino & Al-Nasser 2020; Thamer & Zine 2023) have attracted increased research interest. The pdf and cdf of the THPLD, with shape parameters θ, α and β are defined as follows:

$$f(x;\theta,\alpha,\beta) = \frac{\theta^2}{\beta\theta+\alpha}(\beta+\alpha x)e^{-\theta x} \textit{ and}$$

$$F(x;\theta,\alpha,\beta) = 1-\left(\frac{\beta\theta+\alpha+\alpha\theta x}{\beta\theta+\alpha}\right)e^{-\theta x},$$

where x, θ, $\alpha > 0$ and $\beta\,\theta + \alpha > 0$. Note that the pdf of the THPLD may not satisfy the regularization condition of probability. To solve this issue, here we use the parameter space $\Omega = \{\beta{:}\beta > -\alpha / (1+\theta)\}$ to replace the parameter space $\Omega' = \{\beta{:}\beta > -\alpha / \theta\}$ of β in Shanker et al. (2017).

To obtain a new more flexible and adaptable model, we want to extend the three-parameter Lindley distribution by using the transmutation map method. In this paper, several important mathematical properties and estimation methods are presented.

The paper is structured as follows: In the next section, a new flexible model is proposed. Structural properties such as moments and associated measures, order statistics, and reliability and hazard rate functions are derived subsequently. In the section that follows, the Renyi entropy in information theory is calculated. Next, the maximum likelihood method is used to investigate the parameter estimation. After that, a simulation study is conducted to evaluate the proposed model. Subsequently, a real data example is used to demonstrate the application of the new model. Finally, the conclusions are presented in the last section.

THE TRANSMUTED THREE-PARAMETER LINDLEY DISTRIBUTION

A random variable X is said to have a transmuted three-parameter Lindley distribution (TTHPLD) if its cdf is given by

$$F(x) = (1+\lambda)\left[1-\frac{(\alpha+\beta\theta+\alpha\theta x)e^{-\theta x}}{\alpha+\beta\theta}\right]-\lambda\left[1-\frac{(\alpha+\beta\theta+\alpha\theta x)e^{-\theta x}}{\alpha+\beta\theta}\right]^2$$

$$=\frac{e^{-2\theta x}\left[(e^{\theta x}-1)\beta\theta+\alpha(e^{\theta x}-\theta x-1)\right]\left[e^{\theta x}(\alpha+\beta\theta)+(\alpha+\beta\theta+\alpha\theta x)\lambda\right]}{(\alpha+\beta\theta)^2},$$

and the corresponding pdf of TTHPLD is given by

$$f(x)=\frac{\theta^2}{\beta\theta+\alpha}(\beta+\alpha x)e^{-\theta x}\left[1+\lambda-2\lambda\left(1-\frac{\beta\theta+\alpha+\alpha\theta x}{\beta\theta+\alpha}e^{-\theta x}\right)\right].$$

where α, β, θ and λ are unknown parameters of the new distribution. Moreover, α, β and λ are shape parameters, and θ is the scale parameter. For $\lambda = 0$ and $\beta = 1$, TTHPLD reduces to the three-parameter Lindley distribution (THPLD) and the transmuted two-parameter Lindley distribution (TTPLD), respectively. Moreover, for $\lambda = 0$ and $\beta = 1$, TTHPLD reduces to the two-parameter Lindley distribution (TPLD). For $\lambda = 0$ and α, $\beta \rightarrow 1$, TTHPLD tends to follow the Lindley distribution (LD).

Figures 1 and 2 display the pdf and cdf of the TTHPLD ($\theta,\alpha,\beta,\lambda$) for various values of θ,α,β and λ, respectively.

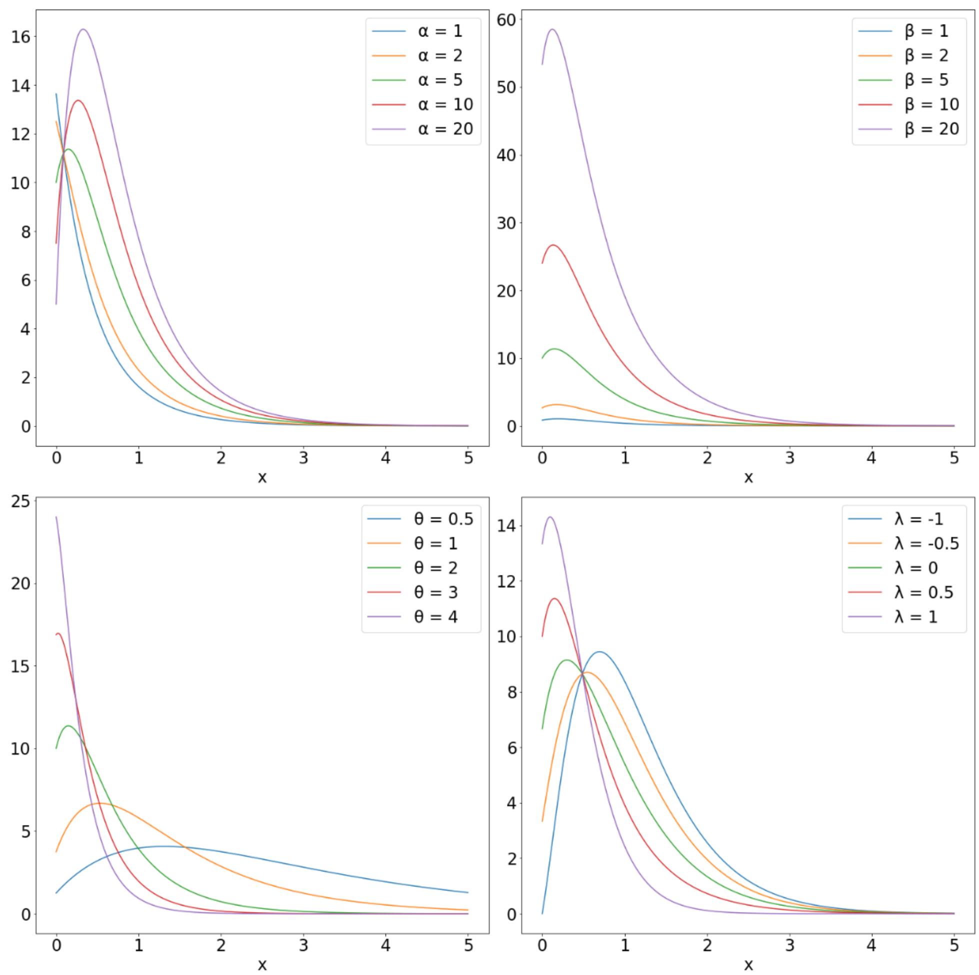

FIGURE 1. The pdf of the TTHPLD for various values of θ, α, β and λ.

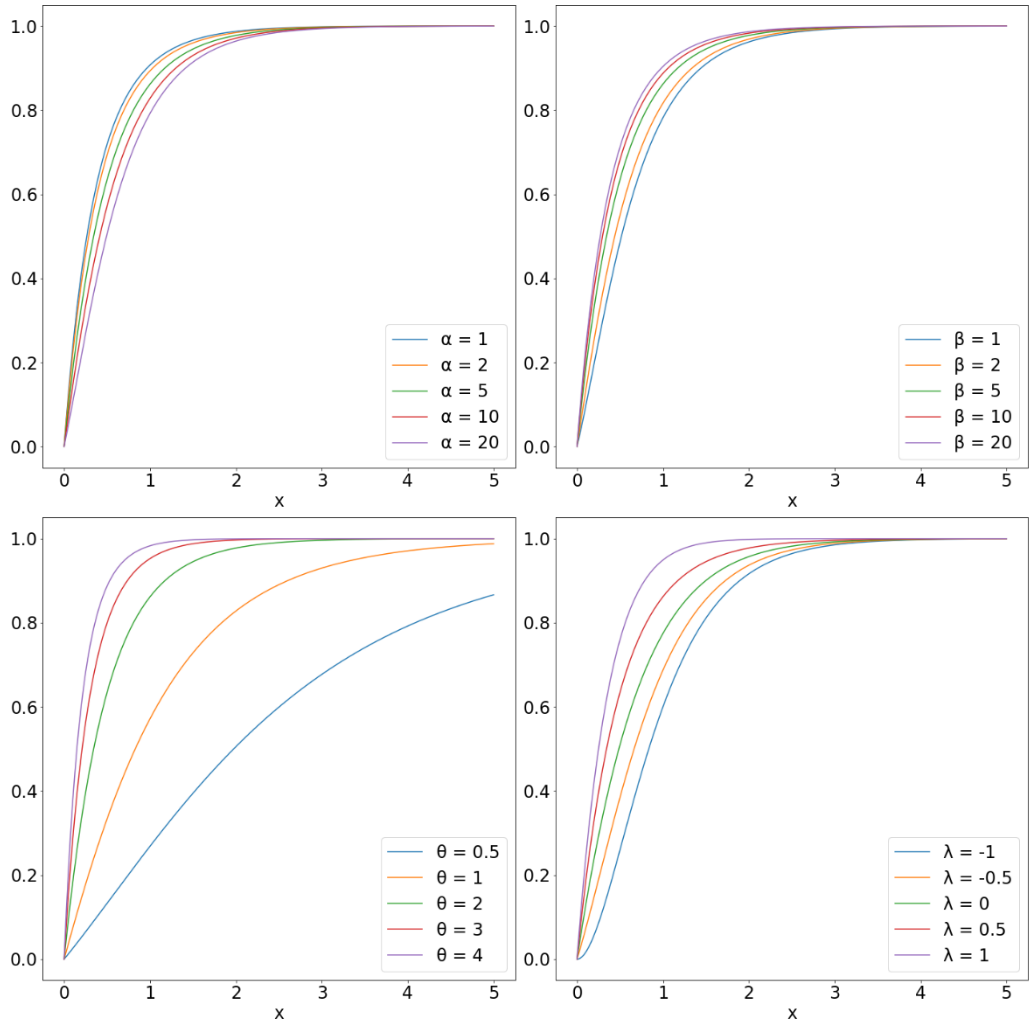

FIGURE 2. The cdf of the TTHPLD for various values of θ, α, β and λ.

STRUCTURAL PROPERTIES OF THE PROPOSED DISTRIBUTION

MOMENTS AND ASSOCIATED MEASURES

In this section, the moment generating function, r-th moment, coefficient of variation, skewness and kurtosis are derived.

Theorem 1 The moment generating function of the TTHPLD is obtained by

$$M_X(t)=\frac{\theta^2}{\beta\theta+\alpha}\left(\frac{\beta-\lambda\beta}{\theta-t}+\frac{\alpha-\lambda\alpha}{(\theta-t)^2}+\frac{2\lambda\beta}{(2\theta-t)}+\frac{4\lambda\theta\alpha\beta+2\lambda\alpha^2}{(2\theta-t)^2}+\frac{4\lambda\theta\alpha^2}{(2\theta-t)^3}\right).$$

Proof $M_X(t)=\int_0^\infty e^{tx}f(x)dx$

$$=\int_0^\infty \frac{\theta^2 e^{tx}}{\beta\theta+\alpha}\Bigg\{(1-\lambda)\beta e^{-\theta x}+(1-\lambda)\alpha x e^{-\theta x}+ 2\beta\lambda e^{-2\theta x}+2\alpha\lambda\left(\frac{2\beta\theta+\alpha}{\beta\theta+\alpha}\right)xe^{-2\theta x} +\frac{2\lambda\theta\alpha^2}{\beta\theta+\alpha}x^2e^{-2\theta x}\Bigg\}dx$$

$$=\frac{\theta^2}{\beta\theta+\alpha}\Bigg\{.(1-\lambda)\beta\int_0^\infty e^{-(\theta-t)x}dx+\alpha(1-\lambda)\int_0^\infty xe^{-(\theta-t)x}dx+2\lambda\beta\int_0^\infty e^{-(2\theta-t)x}dx +2\alpha\lambda\frac{2\beta\theta+\alpha}{\beta\theta+\alpha}\int_0^\infty xe^{-(2\theta-t)x}dx+\frac{2\lambda\theta\alpha^2}{\beta\theta+\alpha}\int_0^\infty x^2e^{-(2\theta-t)x}dx\Bigg\}$$

$$= \frac{\theta^2}{\beta\theta+\alpha}\left(\frac{\beta-\lambda\beta}{\theta-t}+\frac{\alpha-\lambda\alpha}{(\theta-t)^2}+\frac{2\lambda\beta}{(2\theta-t)}+\frac{4\lambda\theta\alpha\beta+2\lambda\alpha^2}{(2\theta-t)^2}+\frac{4\lambda\theta\alpha^2}{(2\theta-t)^3}\right).$$

Hence, the proof is complete.

Theorem 2 The *r*-th moment of the TTHPLD distributed random variable is given as

$$E(X^r) = \frac{\theta^2}{(\beta\theta+\alpha)}\left\{\left(\frac{\beta-\lambda\beta}{\theta^{r+1}}+\frac{2\lambda\beta}{(2\theta)^{r+1}}\right)\Gamma(r+1) + \frac{2\lambda\theta\alpha^2}{(\beta\theta+\alpha)(2\theta)^{r+3}}\Gamma(r+3) + \left(\frac{\alpha-\lambda\alpha}{\theta^{r+2}}+\frac{4\lambda\beta\theta\alpha+2\lambda\alpha^2}{(\beta\theta+\alpha)(2\theta)^{r+2}}\right)\Gamma(r+2)\right\}.$$

Proof. $E(X^r) = \int_0^\infty x^r f(x)dx$

$$= \int_0^\infty x^r \frac{\theta^2}{\beta\theta+\alpha}(\beta+\alpha x)e^{-\theta x}\left(1+\lambda-2\lambda\left(1-\frac{\beta\theta+\alpha+\alpha\theta x}{\beta\theta+\alpha}e^{-\theta x}\right)\right)dx.$$

$$= \frac{\theta^2}{\beta\theta+\alpha}\left\{\int_0^\infty (1-\lambda)x^r\beta e^{-\theta x}dx + \int_0^\infty (\alpha-\lambda\alpha)x^{r+1}e^{-\theta x}dx + 2\lambda\beta\int_0^\infty x^r e^{-2\theta x}dx + 2\alpha\lambda\int_0^\infty \frac{2\beta\theta+\alpha}{\beta\theta+\alpha}x^{r+1}e^{-2\theta x}dx + \int_0^\infty \frac{2\lambda\theta\alpha^2}{\beta\theta+\alpha}x^{r+2}e^{-2\theta x}dx\right\}$$

$$= \theta^2\left\{\left(\frac{\beta-\lambda\beta}{(\beta\theta+\alpha)\theta^{r+1}}+\frac{2\lambda\beta}{(\beta\theta+\alpha)(2\theta)^{r+1}}\right)\Gamma(r+1) + \frac{2\lambda\theta\alpha^2}{(\beta\theta+\alpha)^2(2\theta)^{r+3}}\Gamma(r+3) + \left(\frac{\alpha-\lambda\alpha}{(\beta\theta+\alpha)\theta^{r+2}}+\frac{4\lambda\beta\theta\alpha+2\lambda\alpha^2}{(\beta\theta+\alpha)^2(2\theta)^{r+2}}\right)\Gamma(r+2)\right\}$$

$$= \frac{\theta^2}{(\beta\theta+\alpha)}\left\{\left(\frac{\beta-\lambda\beta}{\theta^{r+1}}+\frac{2\lambda\beta}{(2\theta)^{r+1}}\right)\Gamma(r+1)+\frac{2\lambda\theta\alpha^2}{(\beta\theta+\alpha)(2\theta)^{r+3}}\Gamma(r+3) + \left(\frac{\alpha-\lambda\alpha}{\theta^{r+2}}+\frac{4\lambda\beta\theta\alpha+2\lambda\alpha^2}{(\beta\theta+\alpha)(2\theta)^{r+2}}\right)\Gamma(r+2)\right\}.$$

Hence, the proof is complete.

Simple computations yield the first four raw moments of $X \sim TTHPLD(\theta, \alpha, \beta, \lambda)$ as follows:

$$\mu = E(X) = \frac{\alpha\lambda(5\alpha+4\beta\theta)+8\alpha(1-\lambda)(\alpha+\beta\theta)+2\beta\theta(2-\lambda)(\alpha+\beta\theta)}{4\theta(\alpha+\beta\theta)^2},$$

$$\mu_2 = E(X^2) = \frac{3\alpha\lambda(3\alpha+2\beta\theta)+24\alpha(1-\lambda)(\alpha+\beta\theta)+2\beta\theta(4-3\lambda)(\alpha+\beta\theta)}{4\theta^2(\alpha+\beta\theta)^2},$$

$$\mu_3 = E(X^3) = \frac{3\left(\alpha\lambda(7\alpha+4\beta\theta)+32\alpha(1-\lambda)(\alpha+\beta\theta)+\beta\theta(8-7\lambda)(\alpha+\beta\theta)\right)}{4\theta^3(\alpha+\beta\theta)^2},$$

$$\mu_4 = E(X^4) = \frac{3\cdot\left(5\alpha\lambda(2\alpha+\beta\theta)+80\alpha(1-\lambda)(\alpha+\beta\theta)+\beta\theta(16-15\lambda)(\alpha+\beta\theta)\right)}{2\theta^4(\alpha+\beta\theta)^2},$$

and the variance of X is

$$\sigma^2 = Var(X) = E(X^2) - \left(E(X)\right)^2$$

$$= \frac{4(\alpha+\beta\theta)^2\cdot\left(3\alpha\lambda(3\alpha+2\beta\theta)-24\alpha(\alpha+\beta\theta)(\lambda-1)-2\beta\theta(\alpha+\beta\theta)(3\lambda-4)\right)}{16\theta^2(\alpha+\beta\theta)^4}$$

$$-\frac{\left(-\alpha\lambda(5\alpha+4\beta\theta)+8\alpha(\alpha+\beta\theta)(\lambda-1)+2\beta\theta(\alpha+\beta\theta)(\lambda-2)\right)^2}{16\theta^2(\alpha+\beta\theta)^4}$$

Furthermore, the coefficient of variation (CV), skewness and kurtosis of the distribution work out to

$$CV = \frac{\sigma}{\mu},$$

$$Skewness = \frac{E(X-\mu)^3}{\sigma^3} = \frac{\mu_3 - 3\mu_2\mu + 2\mu^3}{\sigma^3},$$

$$Kurtosis = \frac{E(X-\mu)^4}{\sigma^4} = \frac{\mu_4 - 4\mu_3\mu + 6\mu_2\mu^2 + 3\mu^4}{\sigma^4}.$$

The expressions for CV, skewness and kurtosis are large and complicated; however, their values for different parametric values can be derived and are displayed in Table 1.

According to the data in Table 1, as λ increases, both the mean and variance values consistently decrease. Additionally, the CV and kurtosis follow similar trends, initially increasing and then decreasing. Notably, skewness demonstrates multiple changes as λ increases; it first decreases, then increases, and finally continues to decrease. These findings indicate that the TTHPLD distribution is asymmetric in nature.

ORDER STATISTICS

Order statistics play a vital role in various fields of training and statistical theory, offering a wide range of applications in life testing and reliability analysis.

TABLE 1. Descriptive statistics of TTHPLD with $\theta = 2$, $\alpha = \beta = 3$ for $-1 \le \lambda \le 1$

λ	μ	σ^2	CV	Skewness	Kurtosis
-1	0.9861	0.4651	0.6916	1.4076	-20.1779
-0.9	0.9542	0.4666	0.7159	1.4074	-16.7954
-0.8	0.9222	0.4662	0.7404	1.4167	-13.9072
-0.7	0.8903	0.4637	0.7648	1.4345	-11.4189
-0.6	0.8583	0.4591	0.7894	1.4602	-9.2556
-0.5	0.8264	0.4525	0.8140	1.4930	-7.3570
-0.4	0.7944	0.4439	0.8386	1.5327	-5.6739
-0.3	0.7625	0.4332	0.8632	1.5790	-4.1653
-0.2	0.7306	0.4205	0.8876	1.6317	-2.7970
-0.1	0.6986	0.4057	0.9117	1.6908	-1.5395
0	0.6667	0.3889	0.9354	1.7563	-0.3673
0.1	0.6347	0.3700	0.9584	1.8282	0.7420
0.2	0.6028	0.3492	0.9803	1.9065	1.8085
0.3	0.5708	0.3262	1.0006	1.9908	2.8488
0.4	0.5389	0.3013	1.0185	2.0801	3.8745
0.5	0.5069	0.2743	1.0330	2.1722	4.8866
0.6	0.4750	0.2452	1.0425	2.2620	5.8602
0.7	0.4431	0.2141	1.0444	2.3371	6.7028
0.8	0.4111	0.1810	1.0348	2.3676	7.1251
0.9	0.3792	0.1458	1.0071	2.2718	6.1814
1	0.3472	0.1086	0.9491	1.7815	0.1669

By examining the order statistics, researchers can identify the minimum and maximum values, calculate percentiles, estimate population parameters, assess the accuracy of statistical models, and determine the reliability of systems or products. Let $X_{(1)}, X_{(2)},..., X_{(n)}$ be the order statistics of the random sample $X_1, X_2, ..., X_n$ selected from a pdf and cdf $f(x)$ and $F(x)$, respectively. The pdf of the jth order statistic $X_{(j)}$ is defined as follows:

$$f_{(j)}(x) = \frac{n!}{(j-1)!\,(n-j)!}[F(x)]^{j-1}[1-F(x)]^{n-j}f(x),$$

for $j = 1, 2, ..., n$, hence we have the pdf of the jth TTHPLD random variable $X_{(j)}$ as

$$f_{(j)}(x) = \frac{(\alpha x+\beta)\theta^2 e^{-\theta x} n!}{(\alpha+\beta\theta)(-j+n)!\,(j-1)!}\left(-2\lambda\left(1-\frac{(\alpha\theta x+\alpha+\beta\theta)e^{-\theta x}}{\alpha+\beta\theta}\right)+\lambda+1\right)$$

$$\times\left(-\frac{\left(\alpha(\theta x-e^{\theta x}+1)-\beta\theta(e^{\theta x}-1)\right)\left(\lambda(\alpha\theta x+\alpha+\beta\theta)+(\alpha+\beta\theta)e^{\theta x}\right)}{(\alpha+\beta\theta)^2e^{2\theta x}}\right)^{j-1}$$

$$\times\left(1+\frac{\left(\alpha(\theta x-e^{\theta x}+1)-\beta\theta(e^{\theta x}-1)\right)\left(\lambda(\alpha\theta x+\alpha+\beta\theta)+(\alpha+\beta\theta)e^{\theta x}\right)}{(\alpha+\beta\theta)^2e^{2\theta x}}\right)^{-j+n},$$

where $x > 0$, $\theta > 0$, $\alpha > -\theta$. Moreover, the pdf of the smallest order statistics $X_{(1)} = \min\,\{X_1, X_2, ..., X_n\}$ is given by

$$f_{(1)}(x) = \frac{n\theta^2e^{-2\theta x}}{(\alpha+\beta\theta)^2}(\beta+\alpha x)\left(-2\lambda\left(-\alpha\theta x-\alpha-\beta\theta+(\alpha+\beta\theta)e^{\theta x}\right)+(\alpha+\beta\theta)(\lambda+1)e^{\theta x}\right)$$

$$\times\left(\frac{(\alpha+\beta\theta)^2e^{2\theta x}+\left(\alpha(\theta x-e^{\theta x}+1)+\beta\theta(1-e^{\theta x})\right)\left(\lambda(\alpha\theta x+\alpha+\beta\theta)+(\alpha+\beta\theta)e^{\theta x}\right)}{(\alpha+\beta\theta)^2e^{2\theta x}}\right)^{n-1},$$

and the pdf of the largest order statistics $X_{(n)} = max\,\{X_1, X_2, ..., X_n\}$ has the form as follows:

$$f_{(n)}(x) = \frac{n\theta^2e^{-2\theta x}}{(\alpha+\beta\theta)^2}(\beta+\alpha x)\left(-2\lambda\left(-\alpha\theta x-\alpha-\beta\theta+(\alpha+\beta\theta)e^{\theta x}\right)+\right.$$

$$\left.(\alpha+\beta\theta)(\lambda+1)e^{\theta x}\right)$$

$$\times\left(-\frac{\left(\alpha(\theta x-e^{\theta x}+1)+\beta\theta(1-e^{\theta x})\right)\left(\lambda(\alpha\theta x+\alpha+\beta\theta)+(\alpha+\beta\theta)e^{\theta x}\right)}{(\alpha+\beta\theta)^2e^{2\theta x}}\right)^{n-1}.$$

RELIABILITY ANALYSIS

Reliability and hazard rate functions are fundamental tools in reliability analysis and are widely employed in fields such as finance, engineering, medicine, and insurance. Their applications span various industries and disciplines, enabling informed decision-making, risk assessment, and proactive maintenance strategies.

The reliability function, denoted as $R(t)$, measures the probability of an item or system not failing before a specified time t. Here the reliability function of the TTHPLD is given by

$$R(t) = 1 - F(t) =$$

$$\frac{(\alpha+\beta\theta)^2e^{2t\theta}+\left(\alpha(t\theta-e^{t\theta}+1)+\beta\theta(1-e^{t\theta})\right)\left(\lambda(\alpha t\theta+\alpha+\beta\theta)+(\alpha+\beta\theta)e^{t\theta}\right)}{(\alpha+\beta\theta)^2e^{2t\theta}}.$$

Moreover, the hazard rate function, denoted as $H(t)$, is another important concept in reliability theory. It measures the probability of failure occurring at time t, given that the item has survived until that time. The hazard rate function is also known as the failure rate or the instantaneous failure rate. It quantifies the rate at which failures occur per unit of time and is crucial for understanding the failure characteristics of a system. Here the hazard rate function of the TTHPLD is defined as

$$h(t) = \frac{f_{TTHLD}(t)}{1-F_{TTHLD}(t)}$$

$$= \frac{\theta^2(\alpha t+\beta)\left(-2\lambda\left(-\alpha t\theta-\alpha-\beta\theta+(\alpha+\beta\theta)e^{t\theta}\right)+(\alpha+\beta\theta)(\lambda+1)e^{t\theta}\right)}{(\alpha+\beta\theta)^2e^{2t\theta}+\left(\alpha(t\theta-e^{t\theta}+1)+\beta\theta(1-e^{t\theta})\right)\left(\lambda(\alpha t\theta+\alpha+\beta\theta)+(\alpha+\beta\theta)e^{t\theta}\right)}.$$

Figure 3 displays the shape of the hazard rate function of the TTHPLD when $\alpha = \beta = 2$ and $\theta = 1$ for different values of λ.

RENYI ENTROPY

Renyi entropy is a concept in information theory that extends the notion of entropy to measure the amount of uncertainty or randomness in a probability distribution. A large entropy value indicates greater uncertainty in the data. Unlike Shannon entropy, which is a single value, Renyi entropy is a family of entropy measures parameterized by a parameter, denoted by p. Moreover, the Renyi entropy of a continuous random variable X is defined as

$$I_R(p) = \frac{1}{1-p}log\left(\int_0^\infty f(x)^p dx\right)$$

where $p > 0$ and $p \neq 1$.

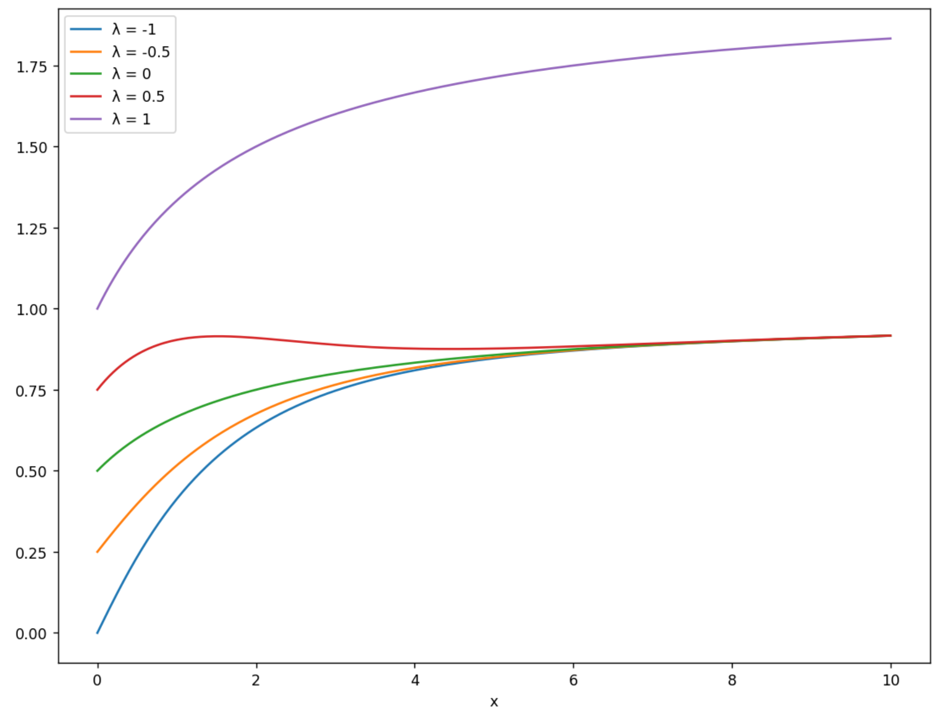

FIGURE 3. The hazard rate function of TTHLD with $\alpha = \beta = 2$, $\theta = 1$ and λ = -1,-0.5,0,0.5,1

Theorem 3 The Renyi entropy of the TTHPLD random variableis defined as

$$I_R(p) = \frac{1}{1-p} log$$

$$\left(\sum_{j=0}^{p} \sum_{i=0}^{p} \sum_{k=0}^{i} (p\,j)(p\,i)(i\,k) \frac{(2\lambda)^i \alpha^{j+k} \beta^{p-j+k} \theta^{2p-j-1}}{(1-\lambda)^{i-p} (\beta\theta+\alpha)^{p+k} (p+i)^{j+k+1}} \Gamma(j + k + 1)\right).$$

Proof. $I_R(p) = \frac{1}{1-p} log \int_0^\infty \left(\left(\frac{\theta^2}{\beta\theta+\alpha}(\beta + \alpha x) e^{-\theta x}\right)\right.$

$$\left.\left((1 + \lambda) - 2\lambda\left(1 - \frac{\beta\theta+\alpha+\alpha\theta x}{\beta\theta+\alpha} e^{-\theta x}\right)\right)\right)^p dx$$

$$= \frac{1}{1-p}\ log\left(\left(\frac{\theta^2}{\beta\theta+\alpha}\right)^p \int_0^\infty (\beta + \alpha x)^p e^{-p\theta x}(1 - \lambda)^p\right.$$

$$\left.\left(1 + \frac{2\lambda}{(1-\lambda)}\left(1 + \frac{\alpha\theta x}{\beta\theta+\alpha}\right)e^{-\theta x}\right)^p dx\right).$$

By using the Binomial theorem,

$$(\beta + \alpha x)^p = \sum_{j=0}^{p} (p\,j)\ \beta^{p-j} (\alpha x)^j,$$

$$\left(1 + \frac{2\lambda}{1-\lambda}\left(1 + \frac{\alpha\theta x}{\beta\theta+\alpha}\right)e^{-\theta x}\right)^p = \sum_{i=0}^{p} (p\,i)\left(\frac{2\lambda}{1-\lambda}\right)^i \left(1 + \frac{\alpha\theta x}{\beta\theta+\alpha}\right)^i e^{-\theta i x}.$$

Hence, we have

$$I_R(p) = \frac{1}{1-p} log\{\left(\frac{\theta^2}{\beta\theta+\alpha}\right)^p (1 - \lambda)^p \int_0^\infty \sum_{j=0}^{p} (p\,j)\left(\beta^{p-j}(\alpha x)^j e^{-p\theta x}\right)$$

$$\times \sum_{i=0}^{p} (p\,i)\left(\frac{2\lambda}{1-\lambda}\right)^i \left(1 + \frac{\alpha\theta x}{\beta\theta+\alpha}\right)^i \left(e^{-\theta x}\right)^i dx$$

$$= \frac{1}{1-p}\ log\left\{\left(\frac{\theta^2}{\beta\theta+\alpha}\right)^p \sum_{j=0}^{p} \sum_{i=0}^{p} \left(\frac{2\lambda}{1-\lambda}\right)^i \left(\frac{p}{j}\right)\left(\frac{p}{i}\right)\right.$$

$$(1 - \lambda)^p \alpha^j \beta^{p-j} \times \int_0^\infty x^j e^{-p\theta x}\left(1 + \frac{\alpha\theta x}{\beta\theta+\alpha}\right)^i \left(e^{-\theta x}\right)^i dx$$

Moreover,

$$\left(1 + \frac{\alpha\theta x}{\beta\theta+\alpha}\right)^i = \sum_{k=0}^{i} (i\,k)\left(\frac{\alpha\theta x}{\beta\theta+\alpha}\right)^k,$$

$$I_R(p) = \frac{1}{1-p} log\left\{\left(\frac{\theta^2}{\beta\theta+\alpha}\right)^p \sum_{j=0}^{p} \sum_{i=0}^{p} \left(\frac{2\lambda}{1-\lambda}\right)^i (p\,j)\ (p\,i)\right.$$

$$(1 - \lambda)^p \alpha^j \beta^{p-j} \times \int_0^\infty x^j e^{-p\theta x} \sum_{k=0}^{i} (i\,k)\left(\frac{\alpha\theta x}{\beta\theta+\alpha}\right)^k \left(e^{-\theta x}\right)^i dx$$

$$= \frac{1}{1-p} log\{\left(\frac{\theta^2}{\beta\theta+\alpha}\right)^p \sum_{j=0}^{p} \sum_{k=0}^{p} \frac{(2\lambda)^i}{(1-\lambda)^{i-p}} (p\, j))\, (p\, i\,)$$

$$(i\, k\,)\alpha^j \beta^{p-j}\left(\frac{\alpha\beta\theta}{\beta\theta+\alpha}\right)^k \int_0^\infty x^{j+k} e^{-\theta x(p+i)} dx\}.$$

Let u = $\theta\,(p + i)\, x$ and $x = \frac{u}{\theta(p+i)}$, , then $du = \theta\,(p + i)\, dx$ and $dx = \frac{du}{\theta(p+i)}$. Since

$$\int_0^\infty x^{j+k} e^{-\theta x(p+i)} dx = \frac{1}{[\theta(p+i)]^{j+k+1}} \int_0^\infty u^{j+k} e^{-u} du$$

$$= \frac{1}{[\theta(p+i)]^{j+k+1}} \Gamma(j + k + 1).$$

Hence, we have

$$I_R(p) = \frac{1}{1-p} log\left(\sum_{j=0}^{p} \sum_{i=0}^{p} \sum_{k=0}^{i} (p\, j\,)(p\, i\,)\right.$$

$$\left.(i\, k\,) \frac{(2\lambda)^i \alpha^{j+k} \beta^{p-j+k} \theta^{2p-j-1}}{(1-\lambda)^{i-p} (\beta\theta+\alpha)^{p+k} (p+i)^{j+k+1}} \Gamma(j + k + 1)\right).$$

Hence, the proof is complete.

MAXIMUM LIKELIHOOD ESTIMATION

In this section, we calculate the parameter estimators for the TTHPLD (θ, α, β, λ). The maximum likelihood estimation method is employed to obtain these estimators ($\hat{\theta}, \hat{\alpha}, \hat{\beta}, \hat{\lambda}$). Let X_1, X_2, ..., X_n be a random sample of size n from the TTHPLD; then the likelihood function is given by

$$L(\theta,\alpha,\beta,\lambda) = \prod_{i=1}^{n} \left(\frac{\theta^2}{\beta\theta+\alpha}(1+\alpha x_i)e^{-\theta x_i}\right)$$

$$\left(1+\lambda-2\lambda\left(1-\frac{\beta\theta+\alpha+\alpha\theta x_i}{\beta\theta+\alpha}e^{-\theta x_i}\right)\right)$$

$$= \left(\frac{\theta^2}{\beta\theta+\alpha}\right)^n \prod_{i=1}^{n} (1+\alpha x_i)e^{-\theta n \sum_{i=1}^{n} x_i} \prod_{i=1}^{n}$$

$$\left(1+\lambda-2\lambda\left(1-\frac{\beta\theta+\alpha+\alpha\theta x_i}{\beta\theta+\alpha}e^{-\theta x_i}\right)\right).$$

Accordingly, the log-likelihood function can be written as

$$l(\theta, \alpha, \beta, \lambda) = log\left\{\left(\frac{\theta^2}{\beta\theta+\alpha}\right)^n \prod_{i=1}^{n}\right.$$

$$\left(1 + \lambda - 2\lambda\left(1 - \frac{\beta\theta+\alpha+\alpha\theta x_i}{\beta\theta+\alpha}e^{-\theta x_i}\right)\right) \times$$

$$\left.\prod_{i=1}^{n} \left(1 + \alpha x_i\right)e^{-\theta n \sum_{i=1}^{n} x_i}\right\}$$

$$= \left\{\sum_{i=1}^{n} log\left(1 + \alpha x_i\right) - n\theta \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} log\right.$$

$$\left(1 + \lambda - 2\lambda\left(1 - e^{-\theta_i x} - \frac{\alpha\theta x_i}{\beta\theta+\alpha}e^{-\theta x_i}\right)\right) +$$

$$n\ log\left(\frac{\theta^2}{\beta\theta+\alpha}\right)$$

The score function is obtained by taking the first partial derivative of the log-likelihood function with respect to θ, α, β and λ, hence

$$\frac{\partial l(\theta,\alpha,\beta,\lambda)}{\partial\theta} = \frac{2n}{\theta} - \frac{n\theta^2}{\beta\theta+\alpha} -$$

$$n\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \frac{2\lambda\left(1+\frac{\alpha\theta x_i-\alpha}{\beta\theta+\alpha}-\frac{\alpha\theta}{(\beta\theta+\alpha)^2}\right)x_i e^{-\theta x_i}}{1-\lambda+2\lambda e^{-\theta x_i}+2\lambda\frac{\alpha\theta x_i}{\beta\theta+\alpha}e^{-\theta x_i}},$$

$$\frac{\partial l(\theta,\alpha,\beta,\lambda)}{\partial\alpha} = -\frac{n}{\beta\theta+\alpha} + \sum_{i=1}^{n} \frac{x_i}{1+\alpha x_i} +$$

$$\sum_{i=1}^{n} \frac{2\lambda\left(\frac{\theta x_i}{\beta\theta+\alpha}e^{-\theta x_i}-\frac{\alpha\theta x_i}{(\beta\theta+\alpha)^2}e^{-\theta x_i}\right)}{1+\lambda-2\lambda\left(1-e^{-\theta x_i}-\frac{\alpha\theta x_i}{\beta\theta+\alpha}e^{-\theta x_i}\right)},$$

$$\frac{\partial l(\theta,\alpha,\beta,\lambda)}{\partial\beta} = -\frac{n\theta}{\beta\theta+\alpha} - \sum_{i=1}^{n}$$

$$\frac{2\alpha\lambda n\theta^2 x_i e^{-\theta x_i}}{(\alpha+\beta\theta)^2\left(-2\lambda\left(-\frac{\alpha\theta x_i e^{-\theta x_i}}{\alpha+\beta\theta}+1-e^{-\theta x_i}\right)+\lambda+1\right)},$$

$$\frac{\partial l(\theta,\alpha,\beta,\lambda)}{\partial\lambda} = \sum_{i=1}^{n} \frac{2e^{-\theta x_i}+2\frac{\alpha\theta x_i}{\beta\theta+\alpha}e^{-\theta x_i}-1}{1+\lambda-2\lambda\left(1-e^{-\theta x_i}-\frac{\alpha\theta x_i}{\beta\theta+\alpha}e^{-\theta x_i}\right)}.$$

The maximum likelihood estimators $\hat{\theta}, \hat{\alpha}, \hat{\beta}$ and $\hat{\lambda}$ can be obtained by setting the score function to zero and solving these equations simultaneously. Since the maximum likelihood equations are nonlinear in nature, the solutions $(\hat{\theta}, \hat{\alpha}, \hat{\beta}, \hat{\lambda})$ have no closed form. These equations can be solved numerically by using numerical optimization algorithms. In this paper, the unknown parameters *θ, α, β* and *λ* are estimated by maximizing by the differential evolution function of Python software. Compared with the Newton-Raphson or quasi-Newton-Raphson methods, which rely on gradient information (e.g., first or second derivatives) from the log-likelihood function, parameter estimation using differential evolution algorithms does not require explicit solutions for derivatives, which renders differential evolution suitable for problems in which computational access to gradient information is difficult or nonexistent.

RANDOM DATA GENERATION AND SIMULATION STUDY

In this section, we conduct a simulation study to generate random variables from the TTHPLD. Subsequently, we estimate the parameters using the maximum likelihood estimation (MLE) method based on the generated sample. To evaluate the accuracy and consistency of the estimates, we calculated the bias and mean squared error (MSE) of the MLE of the parameters. The calculations pertaining to the study were carried out using Python software, version 3.10, with the help of self-programmed codes. The scipy.optimise package in Python software was used to obtain the maximum likelihood estimates of the parameters from TTHPLD.

Using the inversion method, we can generate random numbers from the transmuted three-parameter Lindley distribution via the following equation

$$\frac{e^{-2\theta x}\left[\beta\theta\left(e^{\theta x}-1\right)+\alpha\left(e^{\theta x}-\theta x-1\right)\right]\left[e^{\theta x}(\alpha+\beta\theta)+\lambda(\alpha+\beta\theta+\alpha\theta x)\right]}{(\alpha+\beta\theta)^2}=u,$$

where *u* is a uniformly distributed random variable and *U*(0, 1). Given the sample size *n*, for each u_i, $i = 1, 2, \ldots, n$, we can solve the system of equations for x_i ($i = 1, 2, \ldots, n$) simultaneously. Hence, one can generate random numbers when *α, β, θ* and *λ* are known.

To be more informative, we can assess the performance of MLEs $\hat{\Theta} = (\hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{\lambda})$. Without loss of generality, for different parameter combinations, the TTHPLD corresponding to the selected parameter settings $\Theta = (\alpha, \beta, \theta, \lambda)$ has different shapes. Hence, we consider $\Theta = (\alpha, \beta, \theta, \lambda) = (2.0, 1.5, 1.0, 0.5)$, (1.5, 1.5, 0.5, 0.5), (1.5, 1.0, 1.5, 1.0) and (1.5, 2.0, 0.5, 1.0) for sample sizes *n* = 30, 100, 200 and 500. In each simulation, for a given combination of parameters (*n, α, β, θ, λ*), we first resample the observations 10,000 times from the TTHPLD distribution to obtain the observations $(x_1, x_2, \ldots, x_n)$. Then, we separately calculate the average estimates (AEs), average biases (ABs) and average mean square errors (AMSEs) as follows:

(i) The average estimates (AEs) of the MLEs $\hat{\Theta} = (\hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{\lambda})$ are given by

$$AE(\hat{\theta}) = \frac{1}{N}\sum_{i=1}^{N} \hat{\Theta}_i, i = 1, 2, 3, 4.$$

(ii) The average biases (ABs) of the MLEs $\hat{\Theta} = (\hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{\lambda})$, are given by

$$AB(\hat{\theta}) = \frac{1}{N}\sum_{i=1}^{N} (\hat{\Theta}_i - \Theta_i), i = 1, 2, 3, 4.$$

(iii) The average mean square errors (AMSEs) of the MLEs $\hat{\Theta} = (\hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{\lambda})$, are given by

$$AMSE(\hat{\theta}) = \frac{1}{N}\sum_{i=1}^{N} (\hat{\Theta}_i - \Theta)^2, i = 1, 2, 3, 4.$$

The simulation results are summarized in Table 2.

TABLE 2. Maximum likelihood estimates of the TTHPLD distribution

Sample size	Parameters	$\alpha = 2.0$	$\beta = 1.5$	$\theta = 1.0$	$\lambda = 0.5$
		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$
n=30	AE	1.1778	1.1383	0.6997	0.9971
	AB	0.8222	0.3617	0.2653	-0.5470
	AMSE	0.7629	1.0474	0.0745	0.2625
n=100	AE	1.3220	1.2044	1.0935	0.6743
	AB	0.6781	0.2956	-0.0935	-0.1743
	AMSE	0.5766	0.4283	0.0027	0.0212
n=250	AE	2.1080	1.4184	1.0520	0.5369
	AB	-0.1080	0.0816	-0.0520	-0.0369
	AMSE	0.0084	0.0066	0.0012	0.0041
n=500	AE	2.0058	1.5047	1.0500	0.5309
	AB	-0.0058	-0.0047	-0.0500	-0.0309
	AMSE	0.0002	0.0002	0.0011	0.0009

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Sample size	Parameters	$\alpha = 1.5$	$\beta = 1.5$	$\theta = 0.5$	$\lambda = 0.5$
		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$
n=30	AE	1.2683	1.2038	0.5416	0.4325
	AB	0.2317	0.2962	-0.0416	0.0675
	AMSE	0.6830	0.7316	0.0210	0.1205
n=100	AE	1.2800	1.5753	0.5341	0.4687
	AB	0.2200	-0.0753	-0.0341	0.0313
	AMSE	0.5214	0.0642	0.0103	0.0589
n=250	AE	1.3551	1.5356	0.5198	0.5170
	AB	0.1449	-0.0356	-0.0198	-0.0170
	AMSE	0.3694	0.0297	0.0057	0.0241
n=500	AE	1.4998	1.5013	0.4982	0.5009
	AB	0.0002	-0.0013	0.0018	-0.0009
	AMSE	0.0000	0.0001	0.0001	0.0001
Sample size	Parameters	$\alpha = 1.5$	$\beta = 1.0$	$\theta = 1.5$	$\lambda = 1.0$
		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$
n=30	AE	1.8448	0.5935	1.8646	0.8416
	AB	-0.3448	0.4065	-0.3646	0.1584
	AMSE	0.5796	0.8416	0.6013	0.2863
n=100	AE	1.2819	0.6682	1.7402	0.8840
	AB	0.2181	0.3318	-0.2402	0.1160
	AMSE	0.1436	0.4527	0.1519	0.0872
n=250	AE	1.5828	1.1328	1.6540	0.9478
	AB	-0.0828	-0.1328	-0.1540	0.0522
	AMSE	0.0375	0.0511	0.0557	0.0228
n=500	AE	1.5012	1.0454	1.4917	1.0128
	AB	-0.0012	-0.0454	0.0083	-0.0128
	AMSE	0.0005	0.0293	0.0010	0.0084
Sample size	Parameters	$\alpha = 1.5$	$\beta = 2.0$	$\theta = 0.5$	$\lambda = 1.0$
		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$
n=30	AE	1.0813	1.1236	0.7396	0.5977
	AB	0.4187	0.8764	-0.2396	0.4023
	AMSE	0.3154	0.7461	0.1320	0.3698
n=100	AE	1.7601	1.7650	0.5356	0.8782
	AB	-0.2601	0.2350	-0.0356	0.1218
	AMSE	0.1185	0.1001	0.0067	0.0759
n=250	AE	1.5706	1.8974	0.5050	0.9752
	AB	-0.0706	0.1026	-0.0050	0.0248
	AMSE	0.0431	0.0625	0.0003	0.0144
n=500	AE	1.5047	2.0051	0.4992	1.0103
	AB	-0.0047	-0.0051	0.0008	-0.0103
	AMSE	0.0002	0.0003	0.0001	0.0014

As shown in Table 2, when the sample size n increases, the estimated values of *α, β, θ* and *λ* obtained through the maximum likelihood method converge to the true parameter values. Moreover, the bias and mean square error for each parameter gradually approach 0. Hence, these results illustrate the consistency of the MLEs.

REAL DATA ANALYSIS

In this section, a real dataset is analyzed to demonstrate the adaptability of the TTHPLD. The dataset reported in Al-Khazaleh, Al-Omari and Al-Khazaleh (2016) includes 72 exceedances for the years 1958-1984, rounded to one decimal place of flood peaks (in m^3/s) of the Wheaton River (WR) near Carcross in Yukon Territory, Canada. Here exceedances refer to the specific values by which the flood flow exceeds a predetermined threshold. Thus, these data illustrate the extent to which flood peaks exceeded the set threshold at different times. The data and descriptive statistics are given in Table 3.

From Table 3, the skewness value of the WR data indicates that this dataset is skewed to the left. Smaller values of the Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC) correspond to better distributions, where

$$AIC = -2MLL + 2w, CAIC = -2MLL + \frac{2wn}{n-w-1}, BIC = -2MLL + wLog(n),$$

$$HQIC = 2Log\{Log(n)[w - 2MLL]\},$$

where *w* is the number of parameters; *n* is the sample size and MLL is the maximized log-likelihood. The MLEs, standard deviations (Sds) and 95% confidence intervals (CIs) of the LD, TPLD, TTPLD, THPLD and TTHPLD are presented in Table 4. Note that the parameters of the six models are estimated via the differential evolution method.

Table 5 presents the statistics AIC, CAIC, BIC, HQIC, and -2MLL for the WR data. When comparing the THPLD model to the LD, TPLD, and TTPLD models, it can be observed that the TTHPLD model yields lower AIC, CAIC, BIC, HQIC, and -2MLL values. This indicates that the TTHPLD model performs better than the other four models and provides a better fit for the data.

TABLE 3. 72 exceedances of Wheaton River flood data and associated descriptive statistics

Dataset	0.4	2.2	14.4	20.6	0.7	12	1.9	1.7	13	1.1
	9.3	5.3	11.6	18.7	8.5	14.1	1.1	1.7	2.5	1.4
	14.4	25.5	37.6	22.1	15	2.2	39	11	22.9	1.1
	1.7	0.6	0.1	0.3	0.6	7.3	1.7	7	2.8	9.9
	30	10.4	9	10.7	20.1	3.6	14.1	25.5	30.8	13.3
	3.4	21.5	2.7	27.6	5.6	36.4	4.2	64	11.9	27.1
	2.5	27.4	20.2	5.3	2.5	9.7	1.5	27.5	1	27
	16.8	0.4								

Statistics	Mean	Variance	Median	Skewness	Kurtosis
Values	12.204	151.222	9.5	1.473	5.89

TABLE 4. The MLEs, Sds and 95% confidence intervals of the six models

Model	Parameter	MLE	Sd	Lower limit	Upper limit
LD	$\hat{\theta}$	0.1530	0.0001	0.1320	0.1739
TPLD	$\hat{\theta}$	0.0148	0.0023	-0.0638	0.0935
	$\hat{\alpha}$	0.0921	0.0009	0.0436	0.1406
TTPLD	$\hat{\theta}$	0.0242	0.0010	-0.0272	0.0756
	$\hat{\alpha}$	0.0911	0.0004	0.0569	0.1254
	$\hat{\lambda}$	0.2193	0.0451	0.1435	0.2942
THPLD	$\hat{\theta}$	0.0845	0.0003	0.0580	0.1111
	$\hat{\alpha}$	3.9471	0.0115	3.7711	4.1232
	$\hat{\beta}$	0.0073	0.0036	-0.0920	0.1067
TTHPLD	$\hat{\theta}$	0.0385	0.0061	-0.0897	0.1668
	$\hat{\alpha}$	4.0428	0.0067	3.9084	4.1773
	$\hat{\beta}$	0.1492	0.0096	0.1331	0.1654
	$\hat{\lambda}$	0.0842	0.0031	0.0551	0.1133

TABLE 5. The statistics AIC, CAIC, BIC, HQIC and -2MLL for the WR data

Model	AIC	CAIC	BIC	HQIC	-2MLL
LD	526.4235	523.1469	524.1469	525.5172	264.2117
TPLD	500.3447	493.7914	495.7914	498.5320	252.1723
TTPLD	498.0450	488.2150	491.2150	495.3260	252.0225
THPLD	498.2634	488.4334	491.4334	495.5443	252.1317
TTHPLD	495.9246	482.8180	486.8180	492.2992	251.9623

CONCLUSIONS

A transmuted three-parameter Lindley distribution using the transmutation map method has been introduced. The r-th moment, coefficient of variation, skewness and kurtosis, order statistics, reliability and hazard rate functions and Renyi entropy are derived and studied. Moreover, the maximum likelihood method was used to estimate the parameters of the proposed distribution, and a simulation study was carried out to check the consistency of the MLEs. Furthermore, a real dataset was used to demonstrate the superiority of the new model in data modeling. Finally, the proposed model has at least three advantages. First, the new distribution provides better data fitting ability than that of existing models. Second, the new model corrects the regularization conditions of the three-parameter Lindley distribution, and its more flexible parameter setting extends existing models, such as the Lindley distribution, transmuted Lindley distribution, two-parameter Lindley distribution, transmuted two-parameter Lindley distribution, and three-parameter Lindley distribution. Finally, the proposed model is well suited for datasets where there is a significant right tail or when the tail decays quickly toward zero.

ACKNOWLEDGEMENTS

This work was supported by the Guangdong Basic and Applied Basic Research Foundation (2023A1515110661) and Science and Technology Projects in Guangzhou (2024A04J3550).

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