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QUASI-CONVEXITY AND HERMITE-HADAMARD TYPE INEQUALITIES

(Kekuasicembungan dan Ketaksamaan Jenis Hermite-Hadamard)

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ABSTRACT

In this paper, some inequalities connected with the left and right hand side of Hermite-Hadamard type inequalities are established for functions whose the first and second derivatives absolute values are quasi-convex.

Keywords: Hermite-Hadamard inequality; convex function; quasi-convex function

ABSTRAK

Dalam makalah ini, beberapa ketaksamaan bersabit dengan ketaksamaam jenis Hermite-Hadamard sebelah kiri dan kanan diperoleh untuk fungsi yang nilai mutlak terbitan peringkat pertama dan keduanya adalah fungsi kuasi-cembung.

Kata kunci: ketaksamaan Hermite-Hadamard; fungsi-cembung; fungsi kuasi-cembung

1. Introduction

Let I be an interval of real numbers. A function $f : I \to \mathbb{R}$ is said to be convex on I if the inequality

$$f\left(tx+(1-t)y\right) \le tf(x)+(1-\mathrm{t})f(y) \tag{1}$$

holds for all $x, y \in I$ and $t \in [0,1]$. If the inequality (1) is reversed, then f is said to be concave.

The following double inequality,

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(a)}{2} \tag{2}$$

is well known in the literature as the Hermite-Hadamard integral inequality (see Dragomir & Agarwal 1998), which provides the estimates of the mean value of a continuous convex function defined on an real interval $[a,b]$ with $a<b$. Inequality (2) has attracted many researchers over the years to demostrate new proofs, refinements, extensions and generalization, see for instance Niculescu and Persson (2003), Ciurdariu (2012), Iscan (2013), González *et al*. (2015), Khan *et al.* (2018) and the references cited therein.

Definition 1.1. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be quasi-convex on I if

$$f\left(tx+(1-t)y\right) \le \max\left\{f(x), f(y)\right\} \tag{3}$$

holds for all $x, y \in I$ and $t \in [0,1]$. If f is quasi-concave, then inequality (3) is reversed.

Dragomir and Argawal (1998), Pearce and Pěcarić (2000), Sarikaya and Aktan (2011) provided the inequalities connected with the right part of (3) for functions whose first and second derivatives are convex. Similar results also have been establised by Ion (2007) and Alomari *et al*. (2010) for the class of functions whose derivatives absolute values are quasi-convex, where some of the results are pointed out as follows.

Theorem 1.1. (*Ion* 2007) *Assume* $a,b \in \mathbb{R}$ *with* $a<b$ *and* $f:[a,b] \to \mathbb{R}$ *is a differentiable function on* (a,b), *then we have the inequalities*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \le \begin{cases} \dfrac{(b-a)}{4}\sup\{|f'(a)|,|f'(b)|\} & (4) \\ \qquad \text{for } |f'| \text{ is quasi-convex on } [a,b], & \\ \dfrac{(b-a)}{2(p+1)^{\frac{1}{p}}}\left(\sup\left\{|f'(a)|^{\frac{p}{p-1}},|f'(b)|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}, \ p>1 & (5) \\ \qquad \text{for } |f'|^{\frac{p}{p-1}} \text{ is quasi-convex on } [a,b]. & \end{cases}$$

Theorem 1.2. (*Alomari et al.* 2010) *Let* $f: I \subseteq \mathbb{R} \to \mathbb{R}$ *be a twice differentiable function on the interior points of* I, $a,b \in I$ *with* $a<b$ *and* f'' *be integrable on* $[a,b]$. *Then we have*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right| \le \begin{cases} \dfrac{(b-a)^2}{12}\sup\{|f''(a)|,|f''(b)|\} & (6) \\ \qquad \text{for } |f''| \text{ is quasi-convex on } [a,b], & \\ \dfrac{(b-a)^2}{8}\left(\dfrac{\sqrt{\pi}}{2}\right)^{\frac{1}{p}}\left(\dfrac{\Gamma(1+p)}{\Gamma(\frac{3}{2}+p)}\right)^{\frac{1}{p}} & \\ \quad \times\left(\sup\left\{|f'(a)|^{\frac{p}{p-1}},|f'(b)|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}, \ p>1 & (7) \\ \qquad \text{for } |f''|^{\frac{p}{p-1}} \text{ is quasi-convex on } [a,b]. & \end{cases}$$

Theorem 1.3. (*Alomari et al.* 2010) *Let* $L[a,b]$ *be class of continuous functions defined on* $[a,b]$. *Let* $f: I \subseteq [0,\infty) \to \mathbb{R}$ *be a differentiable function on I such that* $f' \in L[a,b]$, *where* $a,b \in I$ *with* $a<b$. *If* $|f'|^{\frac{p}{p-1}}$, $p>1$ *is quasi-convex on* $[a,b]$. *Then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)}{4(p+1)^{\frac{1}{p}}}\left[\left(\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{\frac{p}{p-1}},\left|f'(b)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}} + \left(\max\left\{\left|f'\left(\frac{a+b}{2}\right)\right|^{\frac{p}{p-1}},\left|f'(a)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right]. \tag{8}$$

The main purpose of this study is to provide some more inequalities related with the Hermite-Hadamard type inequality for twice differentiable functions whose the absolute values of the first and second derivatives are quasi-convex. In particular, we generalise the result of Ion (2007).

2. Some Inequalities of Hermite-Hadamard Type for Quasi-Convex Functions

In order to prove our main results, we need the following integral equalities.

Lemma 2.1. (*Iqbal et al*. 2012) *Let* $f: I\subset\mathbb{R}\to\mathbb{R}$ *be differentiable funtcion on the interior points of* I, *where* $a,b\in I$ *with* $a<b$. *If* $f'\in L[a,b]$ *and* $\mu,\lambda\in[0,\infty)$ *with* $\mu+\lambda\neq 0$, *then*

$$\frac{\mu f(a)+\lambda f(b)}{\mu+\lambda}-\frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{(\mu+\lambda)^2}\left[\int_0^{\mu}(-t)f'\left(\frac{\lambda+t}{\mu+\lambda}a+\frac{\mu-t}{\mu+\lambda}b\right)dt+\int_0^{\lambda}t\, f'\left(\frac{\lambda-t}{\mu+\lambda}a+\frac{\mu+t}{\mu+\lambda}b\right)dt\right] \tag{9}$$

Lemma 2.2. (*Xi & Qi* 2013) *Let* $f: I\subset\mathbb{R}\to\mathbb{R}$ *be a twice differentiable function on the interior points of* I *and* $\lambda\in\mathrm{R}$. *If* $f''\in L[a,b]$ *where* $a,b\in I$ *with* $a<b$, *then*

$$\lambda\frac{f(a)+f(b)}{2}+(1-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx = \frac{(b-a)^2}{16}\int_0^1 t(2\lambda-t)\left[f''\left((1-t)a+t\frac{a+b}{2}\right)+f''\left(t\frac{a+b}{2}+(1-t)b\right)\right]dt. \tag{10}$$

Lemma 2.3. (*Xi & Qi* 2013) *Let* $0\le\lambda\le 1$, $t\in[0,1]$ *and* $r>-1$. *Then,*

$$\int_0^1 t\left|2\lambda-t\right|^r dt = \begin{cases}\dfrac{(2\lambda)^{r+2}+(2\lambda+r+1)(1-2\lambda)^{r+1}}{(r+1)(r+2)}, & 0\le\lambda\le\dfrac{1}{2},\\[2ex] \dfrac{(2\lambda)^{r+2}-(2\lambda+r+1)(2\lambda-1)^{r+1}}{(r+1)(r+2)}, & \dfrac{1}{2}\le\lambda\le 1.\end{cases} \tag{11}$$

We are now in a position to establish some new integral inequalities of Hermite-Hadamard type for twice differentiable and quasi-convex functions.

Theorem 2.1. *Assume* $a,b\in\mathbb{R}$ *with* $a<b$ *and* $f:[a,b]\to\mathbb{R}$ *is a differentiable function on* (a,b). *Then for* $\mu,\lambda\in[0,\infty)$ *with* $\mu+\lambda\neq 0$, *the following inequalities hold true*

$$\left|\frac{\mu f(a)+\lambda f(b)}{\mu+\lambda}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq\begin{cases}\dfrac{(b-a)\left(\mu^2+\lambda^2\right)}{2\left(\mu+\lambda\right)^2}\max\left\{|f'(a)|,|f'(b)|\right\} & (12)\\ \qquad\qquad \text{for } |f'| \text{ is quasi-convex on } [a,b], \\ \dfrac{(b-a)(\mu^2+\lambda^2)}{(\mu+\lambda)^2(p+1)^{1/p}}\left(\max\left\{|f'(a)|^{\frac{p}{p-1}},|f'(b)|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}},\ p>1 & (13)\\ \qquad\qquad \text{for } |f'|^{\frac{p}{p-1}} \text{ is quasi-convex on } [a,b].\end{cases}$$

Proof. By Lemma 2.1 with the fact that $|f'|$ is a quasi-convex function on $[a,b]$ and some elementary integration culculus we have

$$\left|\frac{\mu f(a)+\lambda f(b)}{\mu+\lambda}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq\frac{b-a}{(\mu+\lambda)^2}\left[\int_0^\mu t\left|f'\left(\frac{\lambda+t}{\mu+\lambda}a+\frac{\mu-t}{\mu+\lambda}\right)\right|dt+\int_0^\lambda t\left|f'\frac{\lambda-t}{\mu+\lambda}a+\frac{\mu+t}{\mu+\lambda}b\right|dt\right]$$

$$\leq\frac{b-a}{(\mu+\lambda)^2}\max\left\{|f'(a)|,|f'(b)|\right\}\left[\int_0^\mu t\,dt+\int_0^\lambda t\,dt\right]$$

$$=\frac{b-a}{(\mu+\lambda)^2}\left(\frac{\mu^2+\lambda^2}{2}\right)\max\left\{|f'(a)|,|f'(b)|\right\}.$$

Also, from Lemma 2.1 and applying the weighted Hölder integral inequality, one obtains

$$\left|\frac{\mu f(a)+\lambda f(b)}{\mu+\lambda}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq\frac{b-a}{(\mu+\lambda)^2}\left[\left(\int_0^u t^p dt\right)^{1/p}\left(\int_0^\mu\left|f'\left(\tfrac{\lambda+t}{\lambda+\mu}a+\tfrac{\mu-t}{\lambda+\mu}\right)\right|^q dt\right)^{1/q}\right.$$

$$\left.+\left(\int_0^\lambda t^p dt\right)^{1/p}\left(\int_0^\lambda\left|f'\tfrac{\lambda-t}{\lambda+\mu}a+\tfrac{\mu+t}{\lambda+\mu}b\right|^q dt\right)^{1/q}\right],$$

where $p>1$ with $1/p + 1/q = 1$. Using the fact that $|f'|^q$ is quasi convex function on $[a,b]$, then we have

$$\left|\frac{\mu f(a)+\lambda f(b)}{\mu+\lambda}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\le \frac{b-a}{(\mu+\lambda)^2}\left(\max\left\{|f'(a)|^q,|f'(b)|^q\right\}\right)^{1/q}\left[\left(\int_0^u t^p dt\right)^{1/p}\left(\int_0^\mu dt\right)^{1/q}+\left(\int_0^\lambda t^p dt\right)^{1/p}\left(\int_0^\lambda dt\right)^{1/q}\right]$$
$$=\frac{b-a}{(\mu+\lambda)^2}\left(\max\left\{|f'(a)|^q,|f'(b)|^q\right\}\right)^{1/q}\left[\left(\frac{u^{p+1}}{p+1}\right)^{1/p}u^{1/q}+\left(\frac{\lambda^{p+1}}{p+1}\right)^{1/p}\lambda^{1/q}\right]$$
$$=\frac{(b-a)(\mu^2+\lambda^2)}{(\mu+\lambda)^2(p+1)^{1/p}}\left(\max\left\{|f'(a)|^q,|f'(b)|^q\right\}\right)^{1/q}.$$

The proof of Theorem 2.1 is completed. □

Remark 2.1. If in Theorem 2.1 we choose $\mu=\lambda=1$, then we recapture the Ion's results (4) and (5).

Theorem 2.2. *Assume $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ is a twice differentiable function on $[a,b]$ such that $f''(x)$ is integrable. If $|f''|$ and $|f''|^{\frac{p}{p-1}}\ (p>1)$ are quasi-convex functions on $[a,b]$, then for $0\le\lambda\le1$, the following inequalities hold true:*

$$\left|\lambda\frac{f(a)+f(b)}{2}+(1-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\le\begin{cases}\dfrac{(b-a)^2}{16}H_1(\lambda)\left[\max\left\{|f''(a)|,\left|f''\left(\frac{a+b}{2}\right)\right|\right\}+\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|,|f''(b)|\right\}\right] & (14)\\ \qquad\qquad for\ |f''|\ is\ quasi-convex\ on\ [a,b], & \\ \dfrac{(b-a)^2}{16}\left(H_p(\lambda)\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{p-1}{p}}\times\left[\left(\max\left\{|f''(a)|^{\frac{p}{p-1}},\left|f''\left(\frac{a+b}{2}\right)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right. & \\ \qquad\left.+\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^{\frac{p}{p-1}},|f''(b)|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right],\ p>1 & (15)\\ \qquad\qquad for\ |f''|^{\frac{p}{p-1}},\ is\ quasi-convex\ on\ [a,b], & \end{cases}$$

where

$$H_1(\lambda)=\begin{cases}\dfrac{8\lambda^3-3\lambda+1}{3}, & 0\le\lambda\le\dfrac{1}{2},\\[2mm] \dfrac{3\lambda-1}{3}, & \dfrac{1}{2}\le\lambda\le 1,\end{cases} \tag{16}$$

and

$$H_p(\lambda)=\begin{cases}\dfrac{(2\lambda)^{p+2}+(2\lambda+p+1)(1-2\lambda)^{p+1}}{(p+1)(p+2)}, & 0\le\lambda\le\dfrac{1}{2},\\[2mm] \dfrac{(2\lambda)^{p+2}-(2\lambda+p+1)(2\lambda-1)^{p+1}}{(p+1)(p+2)}, & \dfrac{1}{2}\le\lambda\le 1.\end{cases} \tag{17}$$

Proof: From Lemma 2.2 we have

$$\left|\lambda\frac{f(a)+f(b)}{2}+(1-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\le\frac{(b-a)^2}{16}\int_0^t|t(2\lambda-t)|\left(\left|f''\left((1-t)a+t\frac{a+b}{2}\right)\right|+\left|f''\left(t\frac{a+b}{2}+(1-t)b\right)\right|\right)dt \tag{18}$$

Since $|f''|$ is a quasi convex function on $[a,b]$, thus from (18) we get

$$\left|\lambda\frac{f(a)+f(b)}{2}+(1-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\le\frac{(b-a)^2}{16}\left(\max\left\{|f''(a)|,\left|f''\left(\frac{a+b}{2}\right)\right|\right\}+\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|,|f''(b)|\right\}\right)\int_0^t|t(2\lambda-t)|dt$$
$$\le\frac{(b-a)^2}{16}H_1(\lambda)\left(\max\left\{|f''(a)|,\left|f''\left(\tfrac{a+b}{2}\right)\right|\right\}+\max\left\{\left|f''\left(\tfrac{a+b}{2}\right)\right|,|f''(b)|\right\}\right),$$

where from Lemma 2.3 we have that

$$H_1(\lambda)=\begin{cases}\dfrac{8\lambda^3-3\lambda+1}{3}, & 0\le\lambda\le\dfrac{1}{2},\\[2mm] \dfrac{3\lambda-1}{3}, & \dfrac{1}{2}\le\lambda\le 1.\end{cases}$$

Hence, we obtain the desired result (14).

Agains, from Lemma 2.2 and applying the weighted Hölder's inequality for integrals, we have

$$
\begin{aligned}
&\left|\lambda \frac{f(a)+f(b)}{2}+(1-\lambda) f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\le \frac{(b-a)^2}{16}\left[\left(\int_0^1 t|2\lambda-t|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left|f''\left((1-t)a+t\tfrac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1 t|2\lambda-t|^p dt\right)^{\frac{1}{p}}\left(\int_o^1 t\left|f''\left(t\tfrac{a+b}{2}+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}\right] \\
&=\frac{(b-a)^2}{16}\left(H_p(\lambda)\right)^{\frac{1}{p}}\left[\left(\int_0^1 t\left|f''\left((1-t)a+t\tfrac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_o^1 t\left|f''\left(t\tfrac{a+b}{2}+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}}\right]
\end{aligned}
$$

for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $H_p(\lambda)=\int_0^1 t|2\lambda-t|^p dt$. Therefore, by the quasi-convexity of $|f''|^q$, we obtain

$$
\begin{aligned}
&\left|\lambda \frac{f(a)+f(b)}{2}+(1-\lambda) f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\le \frac{(b-a)^2}{16}\left(H_p(\lambda)\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left[\left(\max\left\{|f''(a)|^q,\left|f''\left(\tfrac{a+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\max\left\{|f''(\tfrac{a+b}{2})|^q,|f''(b)|^q\right\}\right)^{\frac{1}{q}}\right] \\
&=\frac{(b-a)^2}{16}\left(H_p(\lambda)\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{p-1}{p}}\left[\left(\max\left\{|f''(a)|^{\frac{p}{p-1}},\left|f''\left(\tfrac{a+b}{2}\right)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right. \\
&\left.+\left(\max\left\{|f''(\tfrac{a+b}{2})|^{\frac{p}{p-1}},|f''(b)|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right],
\end{aligned}
$$

where $H_p(\lambda)$ as given in (17). Thus, we complete the proof of Theorem 2.2. □

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