Journal of Quality Measurement and Analysis *JQMA* 19(2) 2023, 31-43
e-ISSN: 2600-8602 *http://www.ukm.my/jqma*

SIMPLER ESTIMATORS FOR *k*-FLATED POISSON DISTRIBUTION

(Penganggar-penganggar Mudah bagi Taburan Poisson k-Terinflasi)

RAZIK RIDZUAN MOHD TAJUDDIN

ABSTRACT

The point of inflation or deflation for Poisson distribution cannot be defined objectively. Therefore, the maximum likelihood function may not be used efficiently to derive an estimator. In this paper, two simpler estimators for k-flated Poisson distribution, named as ratio of frequency (RFE) and probability estimators (PE) were developed and discussed. The estimators are based on the position of 'flation', $k = 0, 1, 2, 3$. A comprehensive simulation study was conducted to investigate the unbiasedness and the consistency properties of the estimators. The simulation studies concluded that the estimators are asymptotically unbiased and consistent except for a special case of the RFE, known as the jump RFE, which is only asymptotically unbiased but not consistent. Model fittings on two real datasets showed that the PE is a better estimator than RFE.

Keywords: probability estimator; ratio of frequency estimator

ABSTRAK

Titik inflasi atau deflasi bagi taburan Poisson tidak boleh dikenal pasti secara objektif. Oleh yang demikian, fungsi kebolehjadian maksimum mungkin tidak dapat digunakan secara efisien untuk menerbitkan rumus bagi penganggar. Dalam artikel ini, dua penganggar mudah bagi taburan Poisson k-terinflasi, yang dinamakan sebagai penganggar nisbah kekerapan (RFE) dan penganggar kebarangkalian (PE) telah dibangunkan dan dibincangkan. Penganggar-penganggar ini adalah berasaskan kedudukan 'flasi', $k = 0, 1, 2, 3$. Satu kajian simulasi komprehensif dijalankan bagi mengkaji sifat-sifat ketidakpincangan dan konsisten penganggar-penganggar ini. Kajian simulasi merumuskan bahawa penganggar-penganggar ini bersifat tidak pincang secara asimptot dan konsisten kecuali RFE lompat, satu kes istimewa bagi RFE yang hanya bersifat tidak pincang secara asimptot tetapi tidak konsisten. Pemodelan kepada dua data sebenar menunjukkan PE sebagai penganggar lebih baik daripada RFE.

Kata kunci: penganggar kebarangkalian; penganggar nisbah kebarangkalian

1. Introduction

The term 'flated' was first coined to explain either surplus mass (inflation) or shortage mass (deflation) at a certain point (Böhning & Ogden 2021). Let Y be a random variable that follows a k-flated Poisson distribution with 'flating' parameter ω and rate parameter λ, denoted as $Y \sim kIP(\omega, \lambda)$. The probability mass function for Y is given as

$$\Pr(Y = y \mid \omega, \lambda) \begin{cases} \omega + (1-\omega)\lambda^k \exp(-\lambda)/k! & ; y = k \\ (1-\omega)\lambda^y \exp(-\lambda)/y! & ; y \neq k \end{cases} \tag{1}$$

The distribution can be either inflated or deflated at $y = k$ depending on the value of ω. Generally, when $\omega \in \left(-\lambda^k \exp(-\lambda)/\left(k! - \lambda^k \exp(-\lambda)\right), 0\right)$, the distribution is said to be deflated at k whereas $\omega \in (0,1)$, the distribution is said to be inflated at k. When $\omega = 0$, the distribution is neither inflated nor deflated at k but reduce to the ordinary Poisson distribution. Table 1 below shows the cutoff range for ω for several values of k.

Table 1: The deflation and inflation cutoffs for $k = 0, 1, 2, 3$

k	Deflation cutoff	Inflation cutoff
0	$\left(-\frac{\exp(-\lambda)}{1-\exp(-\lambda)}, 0\right)$	(0,1)
1	$\left(-\frac{\lambda\exp(-\lambda)}{1-\lambda\exp(-\lambda)}, 0\right)$	(0,1)
2	$\left(-\frac{\lambda^2\exp(-\lambda)}{2-\lambda^2\exp(-\lambda)}, 0\right)$	(0,1)
3	$\left(-\frac{\lambda^3\exp(-\lambda)}{6-\lambda^3\exp(-\lambda)}, 0\right)$	(0,1)

Theorem 1. *Let $Y \sim kIP(\omega, \lambda)$. The n^{th} moment about origin can be written as $E(Y^n) = k^n\omega + (1-\omega)E(X^n)$ where X follows Poisson distribution with parameter λ.*

Proof.

$$
\begin{aligned}
E(Y^n) &= \sum_{y=0}^{\infty} y^n Pr(Y = y|\omega, \lambda) \\
&= k^n\left[\omega + (1-\omega)\frac{\lambda^k\, exp(-\lambda)}{k!}\right] + (1-\omega)\sum_{y=0}^{\infty} y^n \frac{\lambda^y\, exp(-\lambda)}{y!} \\
&\qquad - (1-\omega)k^n\frac{\lambda^k\, exp(-\lambda)}{k!} \\
&= k^n\omega + (1-\omega)\sum_{y=0}^{\infty} y^n \frac{\lambda^y\, exp(-\lambda)}{y!} \\
&= k^n\omega + (1-\omega)\sum_{x=0}^{\infty} x^n \frac{\lambda^x\, exp(-\lambda)}{x!} \\
&= k^n\omega + (1-\omega)E(X^n). \ \square
\end{aligned}
$$

Corollary 1
The first and second moment about the origin are respectively written as:

$$E(Y) = k\omega + (1-\omega)E(X) = k\omega + (1-\omega)\lambda,$$

$$E(Y^2) = k^2\omega + (1-\omega)E(X^2) = k^2\omega + (1-\omega)(\lambda + \lambda^2). \tag{2}$$

Theorem 2. *Let $Y \sim kIP(\omega, \lambda)$. The variance and the index of dispersion for Y are respectively given as:*

$$Var(Y) = (1-\omega)[\omega(\lambda^2 + k^2) + \lambda],$$

$$Disp(Y) = \frac{(1-\omega)[\omega(\lambda^2+k^2)+\lambda]}{\omega(k-\lambda)+\lambda}. \tag{3}$$

Proof. The variance can be obtained by taking $Var(Y) = E(Y^2) - [E(Y)]^2$ and the dispersion index can be obtained by taking $Disp(Y) = Var(Y)/E(Y)$. □

Figure 1 shows the heatmap of the dispersion index for different values of ω and λ when $k = 0, 1, 2, 3$. From Figure 1, the kIP distribution can be either underdispersed or overdispersed depending on the 'flating' positions, 'flating' parameter and rate parameter.

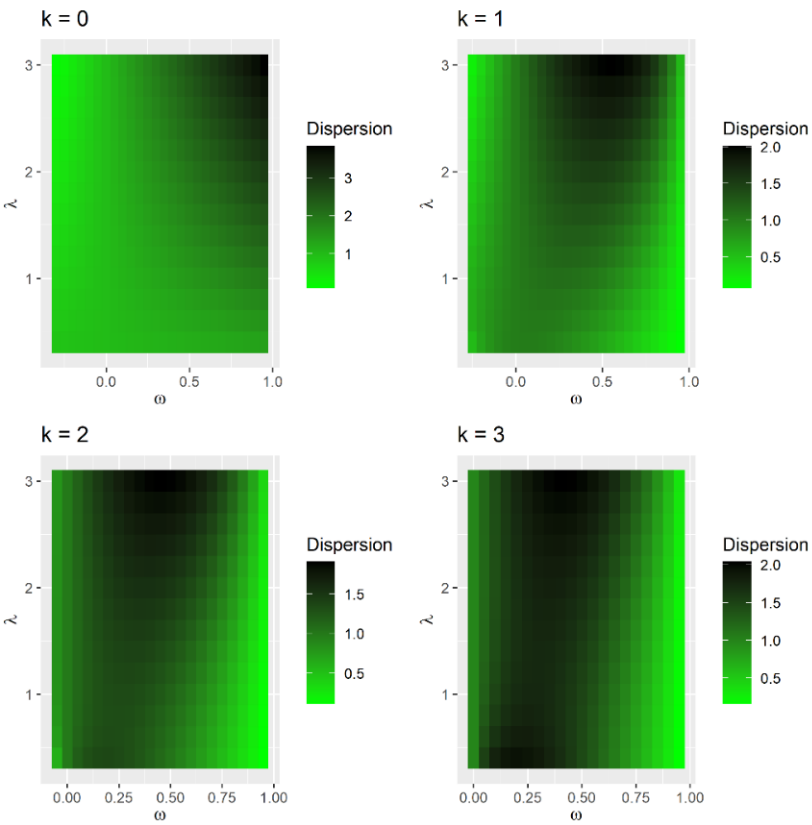

Figure 1: Heatmap of the dispersion index for various values of ω and λ

2. Parameter Estimations

Usually, we focus on the inflating points and ignore the deflating points. However, the 'flating' position cannot be decided objectively. Therefore, two simpler estimation techniques which directly use the sample data were discussed with respect to different 'flating' positions. In this study, we only consider 'flating' positions from 0 to 3 but the idea can be certainly extended to other 'flating' positions. The suggested methods can be solved easily as opposed to solving maximum likelihood equation to obtain the estimator for ω and λ. Notice that, the likelihood function for a k-inflated Poisson distribution is written as:

$$L(\omega,\lambda) = \left[\omega + (1-\omega)\frac{\lambda^k \exp(-\lambda)}{k!}\right]^{n_k} \prod_{y\neq k}^{\infty} \left[(1-\omega)\frac{\lambda^k \exp(-\lambda)}{k!}\right]^{n_y}. \tag{4}$$

From one look, we can realize how difficult the derivation for the estimator will be. To avoid the tedious process of obtaining the formulae for the estimated parameters, the following techniques are presented.

2.1. *Ratio of frequency estimator (RFE)*

As the name suggests, the estimator for parameters is obtained by taking the ratio of two frequencies and equating them to observed frequencies of the data. Three types of RFE named backward RFE, forward RFE and jump RFE are discussed.

2.1.1. *Backward RFE*

The backward RFE compares the ratio of two frequencies, i.e., one at the flation point and another at the previous point. We can write the ratio, $r_{1,k}$ as

$$r_{1,k} = \frac{\Pr\left(Y = k \mid \omega, \lambda\right)}{\Pr\left(Y = k-1 \mid \omega, \lambda\right)} = \frac{\omega\left(k-1\right)!}{\left(1-\omega\right)\lambda^{k-1}\exp\left(-\lambda\right)} + \frac{\lambda}{k}, \tag{5}$$

and compare them with the observed values n_k/n_{k-1} where n_k is the number of data when $Y = k$.

2.1.2. *Forward RFE*

The forward RFE compares the ratio of two frequencies, i.e. one at the flation point and another at the next point. We can write the ratio, $r_{2,k}$ as

$$r_{2,k} = \frac{\Pr\left(Y = k \mid \omega, \lambda\right)}{\Pr\left(Y = k+1 \mid \omega, \lambda\right)} = \frac{\omega\left(k+1\right)!}{\left(1-\omega\right)\lambda^{k+1}\exp\left(-\lambda\right)} + \frac{k+1}{\lambda}, \tag{6}$$

and compare them with the observed values n_k/n_{k+1}.

2.1.3. *Jump RFE*

The jump RFE compares the ratio of two frequencies, i.e., one at the previous point of flation and another at the next point after flation. We can write the ratio, $r_{3,k}$ as

$$r_{3,k} = \frac{\Pr\left(Y = k+1 \mid \omega, \lambda\right)}{\Pr\left(Y = k-1 \mid \omega, \lambda\right)} = \frac{\lambda^2}{k\left(k+1\right)}, \tag{7}$$

and compare them with the observed values n_{k+1}/n_{k-1}.

2.1.4. *RFE for second parameter*

Since there are two parameters to be estimated, another equation is required so that the parameters values can be estimated. For that, the sample mean is equated with the theoretical mean.

$$\bar{y} = k\tilde{\omega} + \left(1-\tilde{\omega}\right)\tilde{\lambda}, \tag{8}$$

where $\tilde{\omega}$ and $\tilde{\lambda}$ are the estimated parameters for ω and λ respectively.

2.1.5. *RFE for different values of k*

Recall that the RFEs depend on the the position of flation, k. The RFEs are tabulated for the first four 'flating' points ($k = 0, 1, 2, 3$). The values of $\tilde{\omega}$ and $\tilde{\lambda}$ can be obtained by solving the two equations in Table 1.

Table 2: Several estimating equations for the two parameters for backward, previous and jump RFE at different 'flating' positions.

RFE	k	Equation 1	Equation 2
Backward	0	Undefined	$\bar{y} = (1-\tilde{\omega})\tilde{\lambda}$
	1	$\frac{\tilde{\omega}}{(1-\tilde{\omega})\exp(-\tilde{\lambda})} + \tilde{\lambda} = \frac{n_1}{n_0}$	$\bar{y} = \tilde{\omega} + (1-\tilde{\omega})\tilde{\lambda}$
	2	$\frac{\tilde{\omega}}{(1-\tilde{\omega})\tilde{\lambda}\exp(-\tilde{\lambda})} + \frac{\tilde{\lambda}}{2} = \frac{n_2}{n_1}$	$\bar{y} = 2\tilde{\omega} + (1-\tilde{\omega})\tilde{\lambda}$
	3	$\frac{2\tilde{\omega}}{(1-\tilde{\omega})\tilde{\lambda}^2\exp(-\tilde{\lambda})} + \frac{\tilde{\lambda}}{3} = \frac{n_3}{n_2}$	$\bar{y} = 3\tilde{\omega} + (1-\tilde{\omega})\tilde{\lambda}$
Forward	0	$\frac{\tilde{\omega}}{(1-\tilde{\omega})\tilde{\lambda}\exp(-\tilde{\lambda})} + \frac{1}{\tilde{\lambda}} = \frac{n_0}{n_1}$	$\bar{y} = (1-\tilde{\omega})\tilde{\lambda}$
	1	$\frac{2\tilde{\omega}}{(1-\tilde{\omega})\tilde{\lambda}^2\exp(-\tilde{\lambda})} + \frac{2}{\tilde{\lambda}} = \frac{n_1}{n_2}$	$\bar{y} = \tilde{\omega} + (1-\tilde{\omega})\tilde{\lambda}$
	2	$\frac{6\tilde{\omega}}{(1-\tilde{\omega})\tilde{\lambda}^3\exp(-\tilde{\lambda})} + \frac{3}{\tilde{\lambda}} = \frac{n_2}{n_3}$	$\bar{y} = 2\tilde{\omega} + (1-\tilde{\omega})\tilde{\lambda}$
	3	$\frac{24\tilde{\omega}}{(1-\tilde{\omega})\tilde{\lambda}^4\exp(-\tilde{\lambda})} + \frac{4}{\tilde{\lambda}} = \frac{n_3}{n_4}$	$\bar{y} = 3\tilde{\omega} + (1-\tilde{\omega})\tilde{\lambda}$
Jump	0	Undefined	$\bar{y} = (1-\tilde{\omega})\tilde{\lambda}$
	1	$\tilde{\lambda} = \sqrt{\frac{2n_2}{n_0}}$	$\bar{y} = \tilde{\omega} + (1-\tilde{\omega})\tilde{\lambda}$
	2	$\tilde{\lambda} = \sqrt{\frac{6n_3}{n_1}}$	$\bar{y} = 2\tilde{\omega} + (1-\tilde{\omega})\tilde{\lambda}$
	3	$\tilde{\lambda} = \sqrt{\frac{24n_4}{n_2}}$	$\bar{y} = 3\tilde{\omega} + (1-\tilde{\omega})\tilde{\lambda}$

2.2. *Probability estimator (PE)*

The PE for parameters ω and λ can be obtained by considering two equations. One equation equates the sample mean with theoretical mean. Another equation equates the sample probability with theoretical probability at 'flating' position. Table 2 lists down several examples of equations for PE.

Table 3: Several estimating equations for the two parameters for backward, previous and jump PE at different 'flating' positions.

k	Equation 1	Equation 2
0	$\frac{n_0}{n} = \tilde{\omega} + (1-\tilde{\omega})\exp(-\tilde{\lambda})$	$\bar{y} = (1-\tilde{\omega})\tilde{\lambda}$
1	$\frac{n_1}{n} = \tilde{\omega} + (1-\tilde{\omega})\tilde{\lambda}\exp(-\tilde{\lambda})$	$\bar{y} = \tilde{\omega} + (1-\tilde{\omega})\tilde{\lambda}$
2	$\frac{n_2}{n} = \tilde{\omega} + (1-\tilde{\omega})\tilde{\lambda}^2\exp(-\tilde{\lambda})/2$	$\bar{y} = 2\tilde{\omega} + (1-\tilde{\omega})\tilde{\lambda}$
3	$\frac{n_3}{n} = \tilde{\omega} + (1-\tilde{\omega})\tilde{\lambda}^3\exp(-\tilde{\lambda})/6$	$\bar{y} = 3\tilde{\omega} + (1-\tilde{\omega})\tilde{\lambda}$

3. Simulation Study

A comprehensive simulation study is conducted to assess the performance of the estimators when $k = 0, 1, 2, 3$ in the aspect of unbiasedness and consistency by utilizing the following formulae:

$$MAD = \frac{1}{M}\sum_{j=1}^{M}\left|\theta - \tilde{\theta}\right|, \quad (9)$$

$$RMSE = \sqrt{\frac{1}{M}\sum_{j=1}^{M}\left(\theta - \tilde{\theta}\right)^2}, \quad (10)$$

where MAD refers to the mean absolute deviation values, RMSE refers to the root mean squared error values and $\tilde{\theta}$ can be either $\tilde{\omega}$ or $\tilde{\lambda}$. For this simulation study, the number of iterations is set at $M = 1000$ and sample sizes are $n = 200\ (200)\ 1400$. The parameters are estimated using R software (R Core Team 2022) and R package 'nleqslv' (Hasselman 2023) to solve the nonlinear functions for the estimation. The R Codes of finding the estimators using R Software and package 'nleqslv' are given in Appendix A.

The results of the simulation studies are presented in Figure 2 – Figure 5. We fix $\omega = 0.1$ and $\lambda = 3.0$ and varies n and k in the simulation study. Since when $k = 0$, backward RFE and the jump RFE are undefined, only the results based on forward RFE and PE are given in Figure 2.

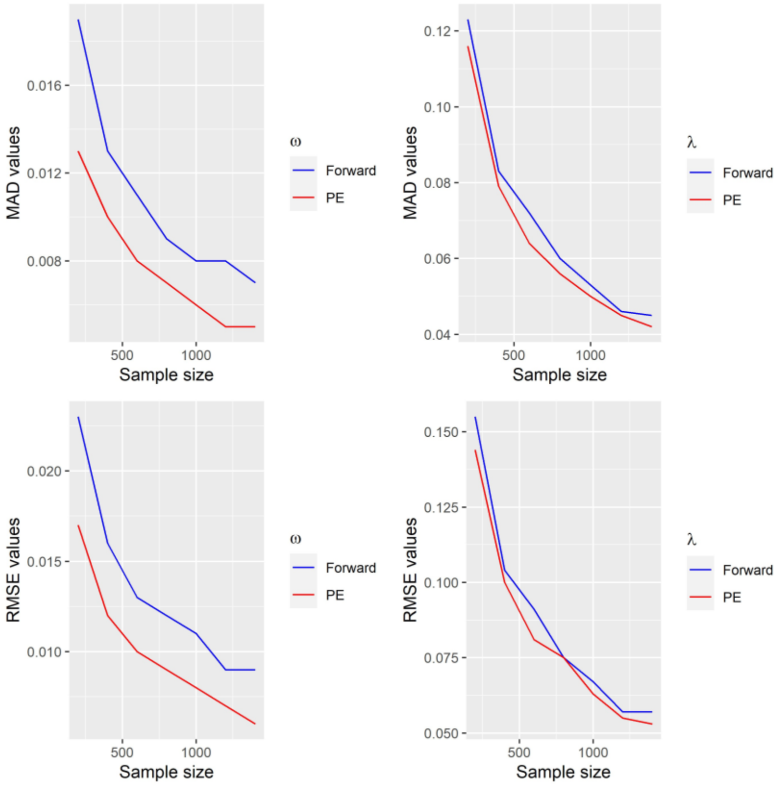

Figure 2: The MAD and RMSE values when $k = 0$ as n increases

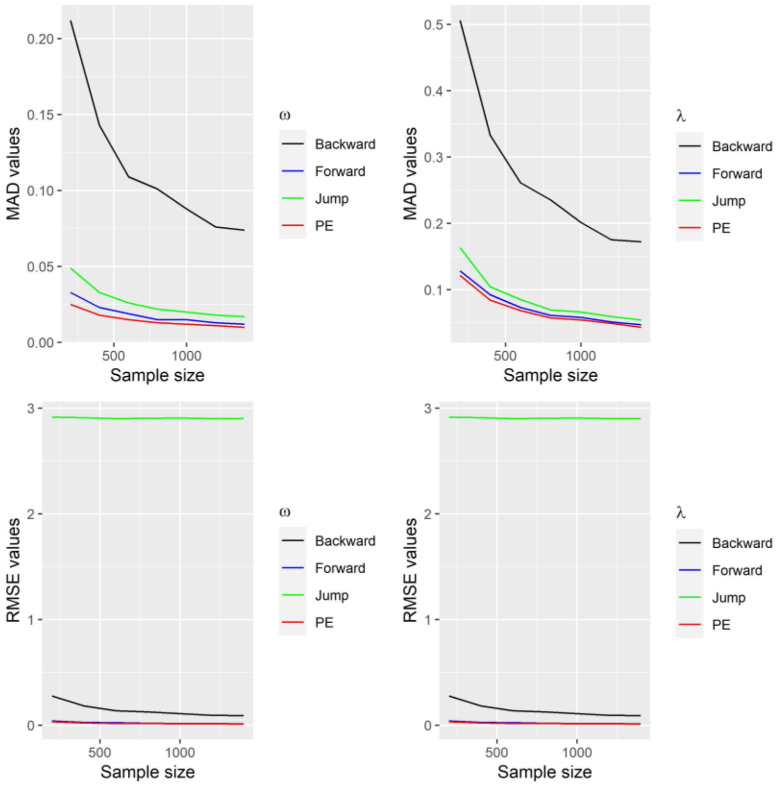

Figure 3: The MAD and RMSE values when $k = 1$ as n increases

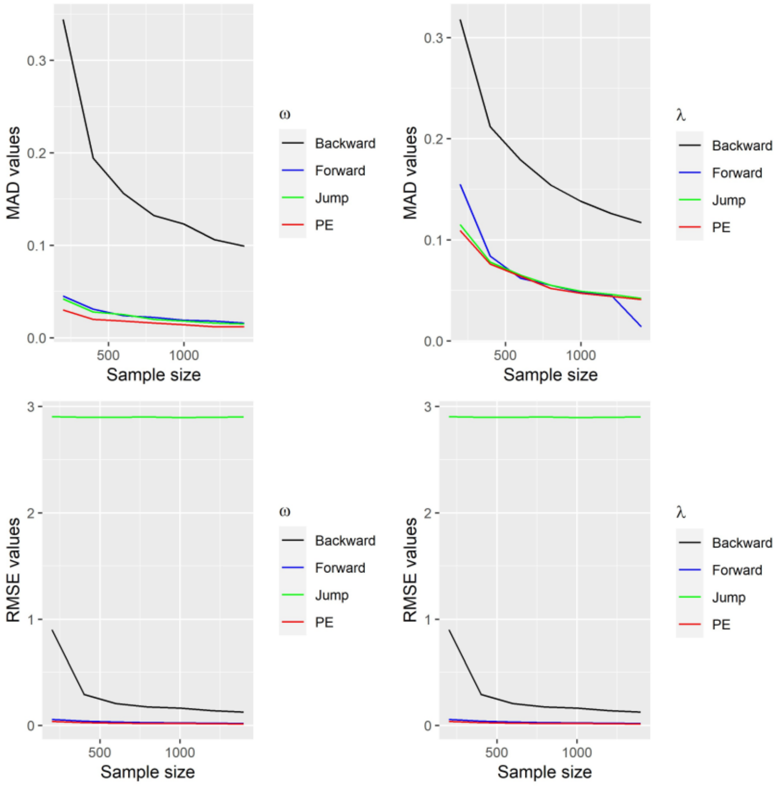

Figure 4: The MAD and RMSE values when $k = 2$ as n increases

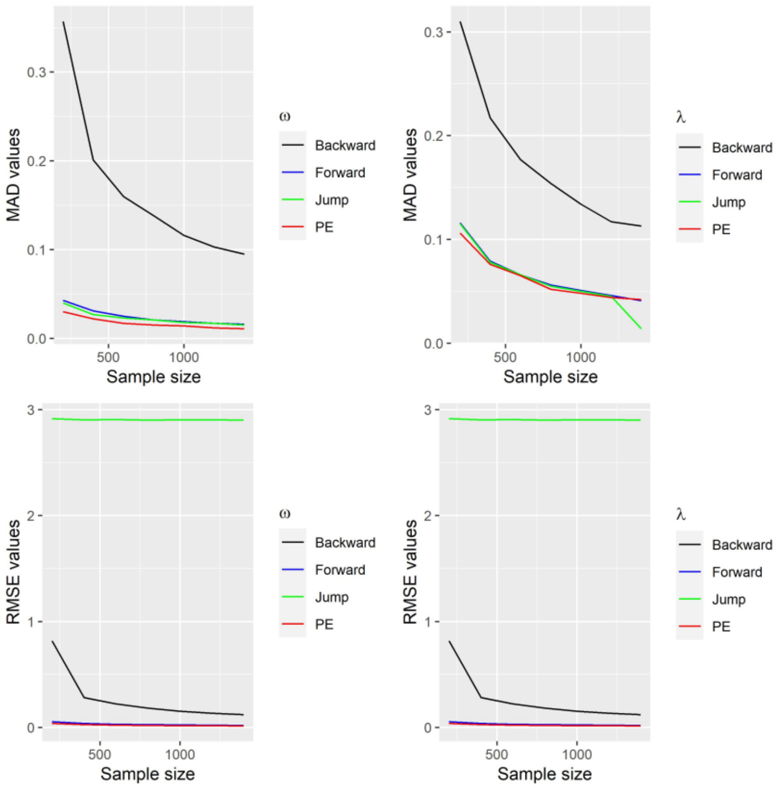

Figure 5: The MAD and RMSE values when $k = 3$ as n increases

From Figure 2 to Figure 5, it can be observed that the MAD values based on the proposed estimators decreases as the sample size increases, indicating that the estimators are asymptotically unbiased. Similar observations regarding the RMSE values can be made except those from jump RFE. The RMSE values for the jump RFE seemed to be constant around 2.900 for every iteration, simulation scenario and sample size. Therefore, the backward RFE, forward RFE and PE are consistent based on decreasing values of RMSE as the sample size increases. For $k \geq 1$, both forward RFE and PE are found to be equally efficient in terms of its MAD and RMSE values, as seen in Figure 3 to Figure 5. From the simulation results, the worst-to-best ranking for the proposed estimators based on their respective MAD and RMSE values for considered k and n is $Jump\ RFE < Backward\ RFE < Forward\ RFE < PE$.

4. Application

Two datasets have been considered in this study: (a) number of death notice of women (80 years of age and over) in the London Times for three consecutive years (Schilling, 1947); (b) number Pap tests in the last 6 years (Lin & Tsai, 2013). Figure 6 shows the histogram of both datasets. For dataset (a), at first glance, it seems like the data is distributed with inflation at position 1 or 2. However, the data may be distributed with deflation at position 0 or 3 as well. Since, we cannot objectively decide the 'flating' positions. Let's consider the 'flating' positions, $k = 0, 1, 2, 3$. For dataset (b), it seems like the data are inflated at $k = 6$. So, for the model fittings, we consider $k = 6$ only.

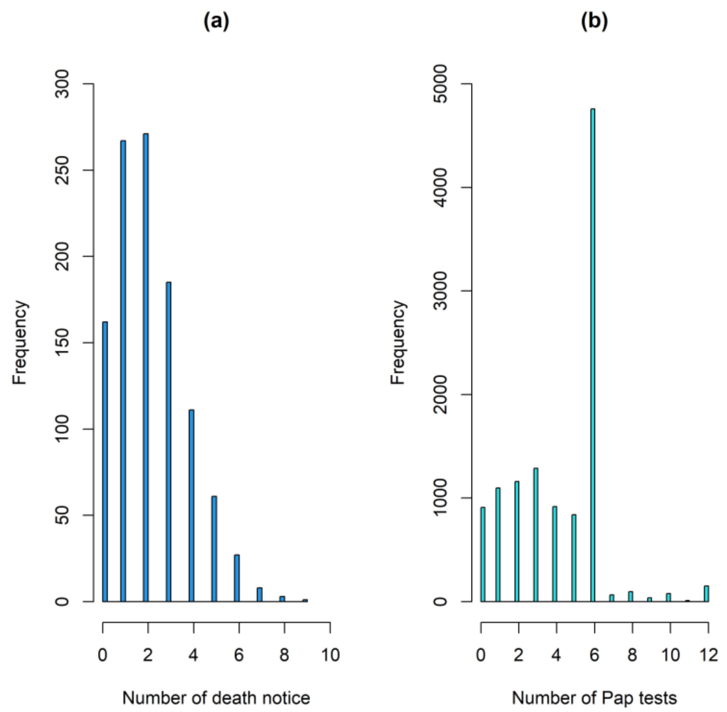

Figure 6: The histogram of (a) death notice data and (b) Pap tests

4.1. ***Model fittings for dataset (a)***

Table 4 shows the estimated parameters for ω and λ using RFE and PE methods for dataset (a). Referring to $\text{sgn}(\widehat{\omega})$ in Table 4, when $k = 0$, the forward RFE and RPE showed that the data are inflated at 0. When $k = 1$, the forward RFE showed that the data are inflated at 1 but the remaining estimator showed that the data are deflated at 1. When $k = 2$, the backward and forward RFE showed that the data are inflated at 2 whereas the remaining estimators supported that the data are deflated at 2. When $k = 3$, estimators other than jump RFE concluded that the data are deflated at 3.

Table 4: The estimated parameters values based on RFE and PE when $k = 0, 1, 2, 3$ for dataset (a)

Dataset (a)	RFE			PE
k	Backward	Forward	Jump	
0	-	$\widehat{\omega} = 0.0368$ $\hat{\lambda} = 2.2393$	-	$\widehat{\omega} = 0.0496$ $\hat{\lambda} = 2.2694$
1	$\widehat{\omega} = -0.0581$ $\hat{\lambda} = 2.0934$	$\widehat{\omega} = 0.0176$ $\hat{\lambda} = 2.1777$	$\widehat{\omega} = -0.3954$ $\hat{\lambda} = 1.8291$	$\widehat{\omega} = -0.0100$ $\hat{\lambda} = 2.1455$
2	$\widehat{\omega} = 0.2161$ $\hat{\lambda} = 2.2002$	$\widehat{\omega} = 0.0142$ $\hat{\lambda} = 2.1592$	$\widehat{\omega} = -3.0295$ $\hat{\lambda} = 2.0389$	$\widehat{\omega} = -0.0040$ $\hat{\lambda} = 2.1523$
3	$\widehat{\omega} = -0.0106$ $\hat{\lambda} = 2.1658$	$\widehat{\omega} = -0.0188$ $\hat{\lambda} = 2.1725$	$\widehat{\omega} = 7.2300$ $\hat{\lambda} = 3.1353$	$\widehat{\omega} = -0.0332$ $\hat{\lambda} = 2.1840$

Table 5 summarizes the model fittings using the RFE and PE in the aspect of root mean-squared error and mean absolute deviation values. The RMSE value using forward RFE is the lowest assuming that the data are flated at 0. However, the MAD value using PE is the lowest under the same assumption. Therefore, both forward RFE and PE estimator assuming the 'flating' point is at 0, can be accepted. To get to the middle ground between smaller RMSE but larger MAD and larger RMSE but smaller MAD, we select the mean of the two values of the estimators such that the new estimated parameters become $\widehat{\omega}' = 0.0432$ and $\hat{\lambda}' = 2.2544$, which yield an RMSE of 11.1835 and an MAD of 8.2405. The resulting estimator gives the smallest RMSE amongst others but second best in MAD values.

Table 5: The estimated parameters values RMSE (MAD) based on RFE and PE when $k = 0, 1, 2, 3$ for dataset (a)

k	RFE			PE
	Backward	Forward	Jump	
0	-	**11.2101** (8.6570)	-	11.5068 (**7.8135**)
1	21.3602 (15.8381)	17.6232 (12.9058)	97.2134 (58.4505)	157.1124 (109.6007)
2	68.8725 (39.2456)	18.6524 (12.7330)	850.2935 (479.2769)	15.8705 (10.9880)
3	16.0501 (11.7476)	15.8315 (11.4318)	2140.8169 (1242.7605)	156.6232 (109.6000)

4.2. ***Model fittings for dataset (b)***

Table 6 shows the estimated parameters for ω and λ using RFE and PE methods for dataset (b) together with their MAD and RMSE values. Referring to $\text{sgn}(\widehat{\omega})$ in Table 6, $\widehat{\omega} > 0$ suggesting that the data are inflated at $k = 6$. Based on the MAD and RMSE values, the model fittings based on PE provide the best fit amongst other contending estimators. Although the values of MAD and RMSE are large, we have to consider the unusual distributions of the number of Pap tests shown in Figure 6 (b). The data refers to the number of Pap tests done in 6 years and the inflation shows at $k = 6$. This might be the results of the same women getting one test each year and thus, contributing to the spike. If we collect a one-year data, there may be inflation at $k = 1$ due to these women or $k = 0$ from women who have not taken the test for that particular year. One may also try model fitting for $k = 3$ as well as $k = 12$.

Table 6: The estimated parameters values based on RFE and PE when $k = 6$.

Dataset (b)	RFE			PE
	Backward	Forward	Jump	
Estimated parameters	$\hat{\omega} = 0.3654$ $\hat{\lambda} = 3.1688$	$\hat{\omega} = 0.4659$ $\hat{\lambda} = 2.6358$	$\hat{\omega} = 0.5704$ $\hat{\lambda} = 1.8177$	$\hat{\omega} = 0.3832$ $\hat{\lambda} = 3.0870$
MAD $(RMSE)$	181.9267 (252.0197)	188.3575 (292.2992)	346.4380 (575.2208)	**162.5721** (**233.8153**)

5. Conclusions

The ‘flating’ positions of the data cannot be identified usually and easily. It depends on the person who are reading the data or viewing the plots related to data. Any changes in the ‘flating’ position will result in different form of probability mass functions, which then further complicate the finding of the estimator based on the maximum likelihood function. This may make the maximum likelihood estimator less desirable due to its complexity in solving.

To combat this issue, two estimation techniques known as the ratio of frequency estimator (RFE) and the probability estimator (PE) were introduced for the ‘flating’ parameter and rate parameter. The RFE is further modified based on the types of ratio, named as backward, forward and jump. These estimators do rely on the ‘flating’ positions, but they can be developed and derived easily.

From the simulation studies, it is found that the PE is the best estimator, followed by forward RFE, backward RFE and jump RFE. The model fittings on the first dataset conclude that PE and forward RFE provide the best fit. However, the model fittings of the second dataset showed that PE is the only estimator that provides a good fit.

Despite being easy to solve the equations, it all depends on the selection of the ‘flating’ positions. Objectively deciding the ‘flating’ positions is a difficult task and relies on the person handling the data. In the case of unclear ‘flating’ positions, it is recommended to consider several ‘flating’ positions and look at the model fittings, which was done in Section 4.1 when handling data with vague ‘flating’ positions.

Acknowledgments

This research receives no funding.

Appendix A.

```
install.packages("nleqslv")
library(nleqslv)

#Generate data
myfun1 = function(n,omega,lambda,k){
  z = rpois(n,lambda)
  pos = sample(n,floor(omega*n))
  z[pos] = k
```

```
  z
}

#Estimate jump RFE
myfunjump = function(n, omega, lambda, k){
  z = myfun1(n, omega, lambda, k)
  lhat = sqrt(k*(k+1)*length(z[z==k+1])/length(z[z==k-1]))
  what = uniroot(function(x) k*x+(1-x)*lhat-mean(z), lower = -100, upper = 1)$root
  c(what, lhat)
}

#Estimate forward RFE
myfunforward = function(n, omega, lambda, k){
  z = myfun1(n, omega, lambda, k)
  eqn = function(x){
    A = (x[1]*factorial(k+1))/((1-x[1])*x[2]^(k+1)*exp(-x[2]))+(k+1)/x[2]-
        length(z[z==k])/length(z[z==k+1])
    B = k*x[1]+(1-x[1])*x[2]-mean(z)
    return(c(A,B))
  }
  nleqslv(c(omega,lambda),eqn)$x
}

#Estimate backward RFE
myfunbackward = function(n,omega,lambda,k){
  z = myfun1(n, omega, lambda, k)
  eqn = function(x){
    A = (x[1]*factorial(k-1))/((1-x[1])*x[2]^(k-1)*exp(-x[2]))+x[2]/k-
        length(z[z==k])/length(z[z==k-1])
    B = k*x[1]+(1-x[1])*x[2]-mean(z)
    return(c(A,B))
  }
  nleqslv(c(omega,lambda),eqn)$x
}

#Estimate PE
myfunPE = function(n,omega,lambda,k){
  z = myfun1(n, omega, lambda, k)
  eqn = function(x){
    A = x[1]+(1-x[1])*(x[2]^k)*exp(-x[2])/factorial(k)-length(z[z==k])/n
    B = k*x[1]+(1-x[1])*x[2]-mean(z)
    return(c(A,B))
  }
  nleqslv(c(omega,lambda),eqn)$x
}
```

References

Böhning D. & Ogden H.E. 2021. General flation models for count data. *Metrika* **84**(2): 245-261.
Hasselman B. 2023. nleqslv: Solve Systems of Nonlinear Equations. R package version 3.3.4. https://cran.r-project.org/web/packages/nleqslv/index.html.
Lin T.H. & Tsai M.H. 2013. Modeling health survey data with excessive zero and *K* responses. *Statistics in Medicine* **32**(9): 1572-1583.
R Core Team. 2022. R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria. https://www.R-project.org/.
Schilling W. 1947. A frequency distribution represented as the sum of two Poisson distributions. *Journal of the American Statistical Association* **42**(239): 407-424.

Department of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
43600 UKM Bangi
Selangor, MALAYSIA.
*Email: rrmt@ukm.edu.my**

Received: 5 April 2023
Accepted: 24 May 2023

*Corresponding author