Journal of Quality Measurement and Analysis *JQMA* 19(3) 2023, 133-142
e-ISSN: 2600-8602 *http://www.ukm.my/jqma*

COMMON FIXED POINT RESULTS FOR BOYD-WONG AND MEIR-KEELER CONTRACTION IN $\mathcal{F}$-METRIC SPACES

(*Hasil Titik Tetap Sepunya untuk Pemetaan Pengecutan Boyd-Wong dan Meir-Keeler dalam Ruang $\mathcal{F}$-Metrik*)

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ABSTRACT

A new notion of metric space generalization has been defined by Jleli and Samet, namely $\mathcal{F}$-metric space, in 2018. The objective of this study is to prove the existence and uniqueness of a common fixed point in the context of $\mathcal{F}$-metric space. We construct theorems of a common fixed point for commuting mapping pairs with Boyd-Wong and Meir-Keeler contraction in this space. Moreover, we extend the results of Park and Bae (1981) and Bera *et al.* (2022) to common fixed point theorems and $\mathcal{F}$-metric space, respectively. The Boyd-Wong contraction is attractive to discuss since we cannot apply the metrizability result on the $\mathcal{F}$-metric space to prove the theorem. The Meir-Keeler contraction is also interesting since it is a generalization of the Boyd-Wong contraction. Lastly, we provide an example of each case to support the findings of our study.

Keywords: $\mathcal{F}$-metric space; commuting mappings; common fixed point; Boyd-Wong contraction; Meir-Keeler contraction

ABSTRAK

Tanggapan baru bagi pengitlakan ruang metrik telah ditakrifkan oleh Jleli dan Samet, iaitu ruang $\mathcal{F}$-metrik, pada tahun 2018. Objektif kajian ini adalah untuk membuktikan kewujudan dan keunikan titik tetap sepunya dalam konteks ruang $\mathcal{F}$-metrik. Kami membina teorem titik tetap sepunya untuk pasangan pemetaan kalis tukar tertib dengan pemetaan pengecutan Boyd-Wong dan Meir-Keeler dalam ruang ini. Selain itu, kami melanjutkan keputusan Park dan Bae (1981) dan Bera *et al.* (2022) kepada teorem titik tetap sepunya dan ruang $\mathcal{F}$-metrik, masing-masing. Pemetaan pengecutan Boyd-Wong menarik untuk dibincangkan kerana kita tidak boleh menggunakan hasil kebolehmetrikan pada ruang $\mathcal{F}$-metrik untuk membuktikan teorem berkenaan. Pemetaan pengecutan Meir-Keeler juga menarik kerana ia merupakan pengitlakan bagi pemetaan pengecutan Boyd-Wong. Akhir sekali, kami menyediakan contoh bagi setiap kes untuk menyokong dapatan kajian.

Kata kunci: ruang $\mathcal{F}$-metrik; pemetaan kalis tukar tertib; titik tetap sepunya; Pemetaan pengecutan Boyd-Wong; Pemetaan pengecutan Meir-Keeler

1. Introduction

After the Banach contraction principle (BCP) was proved by Banach and Cacciopoli, many authors introduced a new type of contraction. Some of them are Boyd and Wong (1969) who introduced nonlinear contraction, namely Boyd-Wong contraction, and Meir and Keeler (1969) who discuss Meir-Keeler contraction. Thus, Gopal *et al.* (2017) introduced the Meir-Keeler contraction as a generalization of the Boyd-Wong contraction. Moreover, the Boyd-Wong and Meir-Keeler contraction class contains the class of Banach contraction (Park & Bae 1981).

On the other hand, Jungck (1976) described the relation between commuting mappings and the common fixed point for contraction. The authors (Karapinar & Agarwal 2022; Gopal &

Bisht 2017) have explained coincidence and common fixed points for several contractive type pairs by changing the contraction condition in Jungck (1976). The existence of a coincidence and common fixed point for contractive mapping pairs is interesting, because it may fail to have even though we have two continuous mappings, that commute, on compact convex sets (Gopal & Bisht 2017).

Furthermore, Park (1978) proved common fixed point theorems for contractive mappings in complete metric space. The author applied the equivalent functional condition to replace the Cauchy condition of a contractive iteration in a complete metric space by Geraghty (1973). Thus, Park and Bae (1981) had an outstanding result in common fixed point theory. They proved the existence of a unique common fixed point that generalized a theorem by Meir and Keeler (1969) in the context of the usual metric space. In addition, Park and Bae (1981) presented the extent of the fixed point theorem for the other contraction, including Boyd-Wong contraction (Boyd & Wong 1969).

Recently, new notions of metric space generalization have been discovered by several authors. One of them is Jleli and Samet (2018) notion which changes the "triangle inequality" in ordinary metric spaces with another form, namely $\mathcal{F}$-metric spaces. Since then, many authors have investigated $\mathcal{F}$-metric space further (Mitrović *et al.* 2019; Alnaser *et al.* 2019; Bera *et al.* 2019; 2022; Asif *et al.* 2019; Som *et al.* 2020; Jahangir *et al.* 2021; Binbasioglu 2021; Altun & Erduran 2022). Jahangir *et al.* (2021) explained the relations of usual metric space with $\mathcal{F}$-metric space. Moreover, the last four years have seen a growing trend towards fixed point (Bera *et al.* 2019; 2022; Jahangir *et al.* 2021) and common fixed point theory (Mitrović *et al.* 2019; Asif *et al.* 2019; Manav & Turkoglu 2019; Binbasioglu 2021) in $\mathcal{F}$-metric spaces. Furthermore, Som *et al.* (2020) proved that Boyd-Wong contractive cannot be obtained with metrizability results. In addition, Bera *et al.* (2022) presented the existence and uniqueness of a fixed point for mapping that satisfies Boyd-Wong type.

In this paper, we extend Bera *et al.* (2022) result by using the Boyd-Wong type notion to ensure the existence and uniqueness of a common fixed point in $\mathcal{F}$-metric space. Previously, Som *et al.* (2020) proved that we could obtain a lot of concepts from usual metric space to $\mathcal{F}$-metric space by using the metrizability result. Despite the Boyd-Wong type contraction can not be obtained from the metrizability result, the research on that type of contraction is still interesting. Consequently, the discussion about the common fixed point theorem for Meir-Keeler contraction in the context of $\mathcal{F}$-metric space is attractive as well. Furthermore, several examples are provided for the two theorems.

2. Preliminaries

Given S and T are two self-mappings on a nonempty set X. If $\eta, \zeta \in X$ exists, such that $T\zeta = S\zeta$ and $\eta = T\eta = S\eta$ then ζ and η is a coincidence point and a common fixed point of T and S, respectively. Let $CP(S,T)$ and $F(S,T)$ represent the set that contains all coincidence points and common fixed points of S and T, respectively. And the mappings T commute with S if only if $TS\eta = ST\eta$ for every $\eta \in X$. In addition, let C_S denote the class of self-mapping T such that $TX \subset SX$ and $TS = ST$.

In 2018, Jleli and Samet Jleli and Samet (2018) defined a certain class $\mathcal{F}$ that contains function $f : (0, +\infty) \to \mathbb{R}$ satisfies two conditions below:

($\mathcal{F}_1$) $f(a) \leq f(b)$, if $0 < a < b$;
($\mathcal{F}_2$) we have $\lim_{b_n \to 0} f(b_n) = -\infty$, for every sequence $(b_n) \subset (0, +\infty)$.

Definition 2.1 (Jleli & Samet 2018)**.** Given $X \neq \emptyset$ and $d_{\mathcal{F}} : X^2 \to [0, \infty)$ is a mapping, then there exists $(g, \alpha) \in \mathcal{F} \times [0, \infty)$ such that for every $(\eta, \theta) \in X^2$ satisfies

(df$_1$) $d_{\mathcal{F}}(\eta, \theta) = 0$ if only if $\eta = \theta$,
(df$_2$) $d_{\mathcal{F}}(\eta, \theta) = d_{\mathcal{F}}(\theta, \eta)$,

(df$_3$) if $\eta \neq \theta$, then for any $N_0 \in \mathbb{N}, N_0 \geq 2$ such that for any $\{v_j\}_{j=1}^{N_0} \subset X$ with $(v_1, v_{N_0}) = (\eta, \theta)$, we have

$$g(d_{\mathcal{F}}(\eta, \theta)) \leq g\left(\sum_{j=1}^{N_0-1} d_{\mathcal{F}}(v_j, v_{j+1})\right) + \alpha.$$

Then, a mapping $d_{\mathcal{F}}$ and the pair $(X, d_{\mathcal{F}})$ are called an $\mathcal{F}$-metric on X and $\mathcal{F}$-metric space, respectively.

Moreover, some concepts on $\mathcal{F}$-metric space presented in Jleli and Samet (2018) as follows

Definition 2.2 (Jleli & Samet 2018)**.** Given sequence $\{\eta_n\}$ is in $\mathcal{F}$-metric space $(X, d_{\mathcal{F}})$.

(i) $\{\eta_n\}$ is $\mathcal{F}$-convergent to $\eta^* \in X$, if $\lim\limits_{n\to\infty} d_{\mathcal{F}}(\eta_n, \eta^*) = 0$.
(ii) $\{\eta_n\}$ is $\mathcal{F}$-Cauchy, if $\lim\limits_{n\to\infty} d_{\mathcal{F}}(\eta_n, \eta_m) = 0$.

The pair $(X, d_{\mathcal{F}})$ is $\mathcal{F}$-complete $\mathcal{F}$-metric space, if each $\mathcal{F}$-Cauchy sequence in X is $\mathcal{F}$-convergent to a point in X.

Proposition 2.3 (Jleli & Samet 2018)**.** Given $(X, d_{\mathcal{F}})$ is an $\mathcal{F}$-metric space and $C \neq \emptyset$ which is a subset of X. Then, the equivalent holds for the following statements.

(a) For any sequence $\{\eta_n\} \subset C$, there exists $\{\eta_{n_k}\} \subset \{\eta_n\}$ and $\eta \in C$ such that

$$\lim_{k\to+\infty} d_{\mathcal{F}}(\eta_{n_k}, \eta) = 0.$$

(b) C is $\mathcal{F}$-compact.

Furthermore, given $(X, d_{\mathcal{F}})$ is an $\mathcal{F}$-metric space with (g, α). Jleli and Samet (2018) defined $d^* : X^2 \to [0, \infty)$ by

$$d^*(\eta, \theta) = \inf\left\{\sum_{i=1}^{N_0-1} d_{\mathcal{F}}(v_i, v_{i+1}) | N_0 \in \mathbb{N}, N_0 \geq 2, \{v_i\}_{i=1}^{N_0} \subseteq X \text{ with } (v_i, v_{N_0}) = (\eta, \theta)\right\}, \tag{1}$$

for any $(\eta, \theta) \in X^2$ and proved that (X, d^*) is a metric space. Then, according to Som *et al.* (2020), $(X, d_{\mathcal{F}})$ is metrizable as regards metric d^*. Based on these results, we establish the following proposition.

Proposition 2.4. Given $(X, d_{\mathcal{F}})$ is an $\mathcal{F}$-metric space with (g, α) and metric d^* defined by (1). Given A and $\{\eta_n\}$ are subsets of X which are a non-empty subset and a sequence, respectively.

(i) $\{\eta_n\}$ is a Cauchy sequence as regards metric d^* if and only if $\{\eta_n\}$ is an $\mathcal{F}$-Cauchy sequence,
(ii) $\{\eta_n\}$ is convergent to $\eta \in X$ as regards metric d^* if and only if $\{\eta_n\}$ is $\mathcal{F}$-convergent to $\eta \in X$,
(iii) X is complete as regards metric d^* if and only if X is $\mathcal{F}$-complete,
(iv) A is compact as regards metric d^* if and only if A is $\mathcal{F}$-compact.

Lastly, we define some contractions for two self-mappings that are commonly used.

Definition 2.5. Given $(X, d_{\mathcal{F}})$ is an $\mathcal{F}$-metric space and S is a continuous self-mappings of X. And T is a self-mapping of X. Then, for every $\eta, \theta \in X$, T is said to be

(a) S-contraction, if $d_{\mathcal{F}}(T\eta, T\theta) \leq \lambda d_{\mathcal{F}}(S\eta, S\theta)$ for some $\lambda \in (0, 1)$,
(b) S-contractive (Park 1978), if $d_{\mathcal{F}}(T\eta, T\theta) < d_{\mathcal{F}}(S\eta, S\theta)$ with $T\eta \neq T\theta$, and
(c) S-nonexpansive, if $d_{\mathcal{F}}(T\eta, T\theta) \leq d_{\mathcal{F}}(S\eta, S\theta)$.

3. Main Result

3.1. *Common fixed point of Boyd-Wong type contraction in $\mathcal{F}$-metric space*

Firstly, we define Boyd-Wong contraction for two self-mapping, namely S and T, in $\mathcal{F}$-metric space with T called ϕ-S-contraction.

Definition 3.1. Given S is a continuous self-mapping of an $\mathcal{F}$-metric space $(X, d_{\mathcal{F}})$ with $(g, \alpha) \in \mathcal{F} \times [0, \infty)$ and ϕ is a self-mapping of $[0, \infty)$ which nondecreasing upper semi-continuous from right such that $\phi(t) < t$ for every $t > 0$ and for any sequence $t_n \to t \geq 0$ implies $\limsup_{n\to\infty} \phi(t_n) \leq \phi(t)$. Any $T \in C_S$ is called an ϕ-S-contraction if for any $\eta, \theta \in X$

$$d_{\mathcal{F}}(T\eta, T\theta) \leq \phi(d_{\mathcal{F}}(S\eta, S\theta)). \tag{2}$$

Secondly, we establish a common fixed point theorem of a self-mappings pair that satisfies ϕ-S-contraction in $\mathcal{F}$-metric space.

Theorem 3.2. *Given $(X, d_{\mathcal{F}})$ is an $\mathcal{F}$-complete $\mathcal{F}$-metric space with $(g, \alpha) \in \mathcal{F} \times [0, \infty)$. Suppose S is a continuous self-mapping of X and $T \in C_S$ is a ϕ-S-contraction. If ϕ satisfies*

$$g(s) > g(\phi(s)) + \alpha, \tag{3}$$

for every $s \in (0, \infty)$, then S and T have a unique common fixed point on X.

Proof. Suppose η_0 is an arbitrary point in X. Since $T(X) \subseteq S(X)$, there exists some $\eta_1 \in X$ such that $S\eta_1 = T\eta_0$. In general, we have a sequence $\{S\eta_n\}$ with $S\eta_n = T\eta_{n-1}$ for any $n \in \mathbb{N}$. By (2), for every $n \in \mathbb{N}$, we obtain

$$\begin{aligned} d_{\mathcal{F}}(S\eta_{n+1}, S\eta_{n+2}) &= d_{\mathcal{F}}(T\eta_n, T\eta_{n+1}) \\ &\leq \phi(d_{\mathcal{F}}(S\eta_n, S\eta_{n+1})) \\ &< d_{\mathcal{F}}(S\eta_n, S\eta_{n+1}). \end{aligned}$$

As a consequence, we have the sequence $\{d_{\mathcal{F}}(S\eta_n, S\eta_{n+1})\}$ which is strictly decreasing and bounded below. Hence, $\lim_{n\to\infty} d_{\mathcal{F}}(S\eta_n, S\eta_{n+1})$ exists, namely p. It is clear that $p \geq 0$. If $p > 0$, by the property of ϕ, we obtain

$$p = \lim_{n\to\infty} d_{\mathcal{F}}(S\eta_n, S\eta_{n+1}) \leq \limsup_{n\to\infty} \phi(d_{\mathcal{F}}(S\eta_{n-1}, S\eta_n)) \leq \phi(p) < p,$$

which is a contradiction. As a result, we have

$$0 = p = \lim_{n\to\infty} d_{\mathcal{F}}(S\eta_n, S\eta_{n+1}). \tag{4}$$

Suppose that $\{S\eta_n\}$ is not an $\mathcal{F}$-Cauchy sequence. Hence, there exists $\epsilon > 0$ and for all $k \in \mathbb{N}$ we defined subsequence $\{S\eta_{m_k}\}$ and $\{S\eta_{n_k}\}$ of $\{S\eta_n\}$ with $m_k \geq n_k \geq k$ and n_k is the smallest number not surpass m_k such that

$$d_{\mathcal{F}}(S\eta_{m_k}, S\eta_{n_k}) \geq \epsilon. \tag{5}$$

Despite the selection of n_k, we have

$$d_{\mathcal{F}}(S\eta_{m_k}, S\eta_{n_k-1}) < \epsilon. \tag{6}$$

Using $(\mathcal{F}_1)$, (df$_3$), (5) and (6), we obtain

$$\begin{aligned} f(\epsilon) &\leq g(d_{\mathcal{F}}(S\eta_{m_k}, S\eta_{m_k+1}) + d_{\mathcal{F}}(S\eta_{m_k+1}, S\eta_{n_k})) + \alpha \\ &\leq g(d_{\mathcal{F}}(S\eta_{m_k}, S\eta_{m_k+1}) + \phi(d_{\mathcal{F}}(S\eta_{m_k}, S\eta_{n_{k-1}}))) + \alpha \\ &\leq g(d_{\mathcal{F}}(S\eta_{m_k}, S\eta_{m_k+1}) + \phi(\epsilon)) + \alpha. \end{aligned}$$

By letting $k \to \infty$, we have

$$g(\epsilon) \leq g(\phi(\epsilon)) + \alpha,$$

which contradicts (3). Therefore, the sequence $\{S\eta_n\}$ is an $\mathcal{F}$-Cauchy sequence. Due to X is $\mathcal{F}$-complete, there is $\eta \in X$ such that

$$\lim_{n\to\infty} d_{\mathcal{F}}(S\eta_n, \eta) = 0. \tag{7}$$

Suppose that $T\eta \neq S\eta$. Thus, by (df$_3$), we obtain

$$g(d_{\mathcal{F}}(T\eta, S\eta)) \leq g(d_{\mathcal{F}}(T\eta, TS\eta_n) + d_{\mathcal{F}}(TS\eta_n, S\eta)) + \alpha, \tag{8}$$

for every $n \in \mathbb{N}$. If there are $S\eta_1$ and $S\eta_2$ such that $d_{\mathcal{F}}(S\eta_1, \eta) = 0 = d_{\mathcal{F}}(S\eta_2, \eta)$, then $S\eta_1 = \eta = S\eta_2$ which is a contradiction. Consequently, we select a sequence $\{S\eta_{n_q}\} \subseteq \{S\eta_n\}$ such that $d_{\mathcal{F}}(S\eta_{n_q}, \eta) \neq 0$ for every $q \in \mathbb{N}$. By the given condition, (8) and $(\mathcal{F}_1)$, we obtain

$$\begin{aligned} g(d_{\mathcal{F}}(T\eta, S\eta)) &\leq g(d_{\mathcal{F}}(T\eta, TS\eta_{n_q}) + d_{\mathcal{F}}(ST\eta_{n_q}, S\eta)) + \alpha \\ &\leq g(\phi(d_{\mathcal{F}}(SS\eta_{n_q}, S\eta)) + d_{\mathcal{F}}(SS\eta_{n_{q+1}}, S\eta)) + \alpha \\ &\leq g(d_{\mathcal{F}}(SS\eta_{n_q}, S\eta) + d_{\mathcal{F}}(SS\eta_{n_{q+1}}, S\eta)) + \alpha. \end{aligned} \tag{9}$$

By the condition $(\mathcal{F}_2)$ and (7), we obtain

$$g(d_{\mathcal{F}}(SS\eta_{n_q}, S\eta) + d_{\mathcal{F}}(SS\eta_{n_{q+1}}, S\eta)) + \alpha \to -\infty,$$

as $q \to \infty$, which contradicts $d_{\mathcal{F}}(T\eta, S\eta) > 0$. Thus, we have $S\eta = T\eta = z$, for $z \in X$. Moreover, suppose that $Sz \neq z$. Since T and S are commuting, then

$$d_{\mathcal{F}}(z, Sz) = d_{\mathcal{F}}(T\eta, ST\eta) = d_{\mathcal{F}}(T\eta, TS\eta) \leq \phi(d_{\mathcal{F}}(S\eta, SS\eta)) < d_{\mathcal{F}}(z, Sz),$$

which is a contradiction. As a consequence, we have $z = Sz = ST\eta = TS\eta = Tz$. In other words, z is a common fixed point of T and S. Now, we prove the uniqueness. Suppose these mappings have the other common fixed point, namely θ with $\theta \neq z$. Thus, we obtain

$$d_{\mathcal{F}}(\theta, z) = d_{\mathcal{F}}(T\theta, Tz) \leq \phi(d_{\mathcal{F}}(S\theta, Sz)) < d_{\mathcal{F}}(\theta, z),$$

which is a contradiction. □

Now, the example is given to support our result.

Example 3.3. Given $X = \mathbb{R}$ and $d_{\mathcal{F}} : X^2 \to \mathbb{R}$ is defined by

$$d_{\mathcal{F}}(\eta, \theta) = \begin{cases} (\eta - \theta)^2, \ \eta, \theta \in [0, 3]; \\ |\eta - \theta|, \ \text{otherwise}. \end{cases}$$

for any $(\eta, \theta) \in X$. It is clear that $d_{\mathcal{F}}$ satisfies (df$_1$) and (df$_2$). Since

$$2 = d_{\mathcal{F}}(2, 4) > d_{\mathcal{F}}\left(2, \frac{5}{2}\right) + d_{\mathcal{F}}\left(\frac{5}{2}, 4\right) = \frac{7}{4},$$

then $d_{\mathcal{F}}$ does not satisfy triangle inequality. However, $d_{\mathcal{F}}$ satisfies (df$_3$) with $g(s) = \ln(s)$, for any $s > 0$, and $\alpha = \ln 3$. Thus, $(X, d_{\mathcal{F}})$ is an $\mathcal{F}$-metric space. Next, we define metric d^* as in Equation (1). obviously, X is complete as regards metric d^* since $X = \mathbb{R}$. Using Proposition 2.4, we obtain that X is $\mathcal{F}$-complete.

Given S and T is a self-mapping on X defined by

$$S\eta = \begin{cases} \frac{\eta+4}{2}, \text{ if } \eta \geq 4; \\ 4, \text{ if } \eta < 4, \end{cases}$$

and $T\eta = 4$ for each $\eta \in X$. We also define a nondecreasing upper semi-continuous from right mapping ϕ by

$$\phi(s) = \frac{1}{4}s, \quad \text{for any } s \in [0, \infty),$$

which satisfies for any sequence $s_n \to s \geq 0$ implies $\limsup_{n\to\infty} \phi(s_n) \leq \phi(s)$, $\phi(s) < s$ and $g(s) > g(\phi(s)) + \alpha$ for every $s > 0$. Let arbitrary points $\eta, \theta \in X$. If $\eta, \theta \in [4, \infty)$, then we obtain

$$d_{\mathcal{F}}(T\eta, T\theta) = 0 \leq \phi(d_{\mathcal{F}}(S\eta, S\theta)) = \frac{1}{8}|\eta - \theta|.$$

Otherwise, we have $d_{\mathcal{F}}(T\eta, T\theta) = 0 \leq \phi(d_{\mathcal{F}}(S\eta, S\theta))$. Hence, T and S satisfy condition (2) for every $\eta, \theta \in \mathbb{R}$. Hence, the conditions of Theorem 3.2 are satisfied. Therefore, the S and T have a unique common fixed point, i.e., $z = 4 = S(4) = T(4)$.

Furthermore, we give the consequences of Theorem 3.2 as follows.

Remark 3.4. By using the same way in Proposition 2.7 in Som *et al.* (2020), if T is a ϕ-S-contraction (2) in the context of $\mathcal{F}$-metric space, then we have T is S-nonexpansive contraction as regards metric d^*. Accordingly, T need not be a ϕ-S-contraction as regards metric d^* if T is a ϕ-S-contraction in $\mathcal{F}$-metric space. We conclude that Theorem 3.2 cannot be proved by the metrizability result.

Remark 3.5. By replacing continuous self-mapping S with identity mapping 1_X, we will get a fixed point theorem for Boyd-Wong contraction in Bera *et al.* (2022). As a result, we consider that Theorem 3.2 is an extension of Theorem 2.1 (Bera *et al.* 2022).

3.2. *Common fixed point of Meir-Keeler type contraction in $\mathcal{F}$-metric space*

The following is the Meir-Keeler contraction definition for self-mappings S and T with $T \in C_S$.

Definition 3.6. Given S is a continuous self-mapping of an $\mathcal{F}$-metric space $(X, d_{\mathcal{F}})$ with $(g, \alpha) \in \mathcal{F} \times [0, \infty)$ and $T \in C_S$. A self-mapping T of X is called an (ϵ, δ)-S-contraction if for any $\epsilon > 0$, there exists $\delta > 0$ such that for all $\eta, \theta \in X$,

(M1) $\epsilon \leq d_{\mathcal{F}}(S\eta, S\theta) < \epsilon + \delta$ implies $d_{\mathcal{F}}(T\eta, T\theta) < \epsilon$,
(M2) $T\eta = T\theta$ whenever $S\eta = S\theta$.

Clearly, (ϵ, δ)-S-contraction implies S-contractive since from (M1),

$$d_{\mathcal{F}}(T\eta, T\theta) < d_{\mathcal{F}}(S\eta, S\theta), \tag{10}$$

whenever $S\eta \neq S\theta$ for all $\eta, \theta \in X$.

Theorem 3.7. *Given $(X, d_{\mathcal{F}})$ is an $\mathcal{F}$-compact $\mathcal{F}$-metric space with $(g, \alpha) \in \mathcal{F} \times [0, \infty)$. Suppose that S is a continuous self-mapping on X and $T \in C_S$ is an (ϵ, δ)-S-contraction, then S and T have a unique common fixed point in X.*

Proof. Suppose $\eta_0 \in X$ is an arbitrary point. Since $T(X) \subseteq S(X)$, we can construct iteration sequence $\{S\eta_n\}$ with $S\eta_n = T\eta_{n-1}$ for every $n \in \mathbb{N}$. If we have $d_{\mathcal{F}}(S\eta_k, S\eta_{k+1}) = 0$ for some $k \in \mathbb{N}$, then $S\eta_k = S\eta_{k+1} = T\eta_k$. In other words, η_k is a coincidence point of S and T. Suppose that $d_{\mathcal{F}}(S\eta_n, S\eta_{n+1}) \neq 0$ for all $n \in \mathbb{N}$ and $\inf\{d_{\mathcal{F}}(S\eta_n, S\eta_{n+1})\} = r$ for some $r > 0$. By (10), we have

$$d_{\mathcal{F}}(S\eta_{n+1}, S\eta_{n+2}) = d_{\mathcal{F}}(T\eta_n, T\eta_{n+1}) < d_{\mathcal{F}}(S\eta_n, S\eta_{n+1}),$$

for every $n \in \mathbb{N}$. Hence, $\{d_{\mathcal{F}}(S\eta_n, S\eta_{n+1})\}$ is a decreasing sequence. Then, we have

$$\lim_{n\to\infty} d_{\mathcal{F}}(S\eta_n, S\eta_{n+1}) = r.$$

By (M1), there exists $\delta > 0$ such that

$$r \leq d_{\mathcal{F}}(S\eta, S\theta) < r + \delta \ \text{ implies } \ d_{\mathcal{F}}(T\eta, T\theta) < r, \tag{11}$$

for every $\eta, \theta \in X$. Since $\lim_{n\to\infty} d_{\mathcal{F}}(S\eta_n, S\eta_{n+1}) = r$, there exists $N_0 \in \mathbb{N}$ such that for every natural number $m \geq N_0$ implies

$$r \leq d_{\mathcal{F}}(S\eta_m, S\eta_{m+1}) < r + \delta. \tag{12}$$

By (11), we have $d_{\mathcal{F}}(S\eta_{m+1}, S\eta_{m+2}) = d_{\mathcal{F}}(T\eta_m, T\eta_{m+1}) < r$, for all $m \geq N_0$, which contradicts (12). Then, we have $\lim_{n\to\infty} d_{\mathcal{F}}(S\eta_n, S\eta_{n+1}) = 0$. Since $(X, d_{\mathcal{F}})$ is an $\mathcal{F}$-compact $\mathcal{F}$-metric space and $\{S\eta_n\} \subset X$, there exists a subsequence $\{S\eta_{n_k}\}$ of $\{S\eta_n\}$ such that

$$\lim_{k\to+\infty} d_{\mathcal{F}}(S\eta_{n_k}, \eta) = 0,$$

for some $\eta \in X$. Since S is continuous and T commutes with S, we have $\{SS\eta_{n_k}\} = \{ST\eta_{n_k-1}\} = \{TS\eta_{n_k-1}\}$ converges to $T\eta$. If we can find some m_k such that $SS\eta_{m_k} = SS\eta_{m_k+1} = SS\eta_{m_k+2} = \dots$, then $\{SS\eta_{n_k}\}$ converges to $SS\eta_{m_k}$ and $SS\eta_{m_k} = SS\eta_{m_k+1} = ST\eta_{m_k} = TS\eta_{m_k}$. We obtain a coincidence point $S\eta_{m_k}$ with $SS\eta_{m_k} = S\eta$.

Suppose $d_{\mathcal{F}}(SS\eta_{n_k}, SS\eta_{n_k+1}) > 0$ for all n_k. Then, for any $\epsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for any $n_k \geq N_0$, $d_{\mathcal{F}}(SS\eta_{n_k}, S\eta) < \frac{\epsilon}{3}$ and we can find $m_k \geq N_0$ such that $SS\eta_{m_k} \neq S\eta$. Assume that $d_{\mathcal{F}}(S\eta, T\eta) > 0$, by (df$_3$) and (10), then we obtain

$$\begin{aligned} g(d_{\mathcal{F}}(S\eta, T\eta)) &\leq g(d_{\mathcal{F}}(S\eta, ST\eta_{n_k}) + d_{\mathcal{F}}(ST\eta_{n_k}, T\eta)) + \alpha \\ &= g(d_{\mathcal{F}}(S\eta, SS\eta_{n_k+1}) + d_{\mathcal{F}}(TS\eta_{n_k}, T\eta)) + \alpha \\ &< g(d_{\mathcal{F}}(S\eta, SS\eta_{n_k+1}) + d_{\mathcal{F}}(SS\eta_{n_k}, S\eta)) + \alpha. \end{aligned}$$

And by $(\mathcal{F}_2)$, then we have

$$g(d_{\mathcal{F}}(S\eta, T\eta)) \leq g(d_{\mathcal{F}}(S\eta, SS\eta_{n_k+1}) + d_{\mathcal{F}}(SS\eta_{n_k}, S\eta)) + \alpha \to -\infty,$$

as $k \to \infty$ which contradicts $d_{\mathcal{F}}(S\eta, T\eta) > 0$. Hence, we have $S\eta = T\eta$. Let $S\eta = T\eta = \zeta$ and suppose $d_{\mathcal{F}}(S\zeta, \zeta) > 0$. By (10), we have

$$d_{\mathcal{F}}(S\zeta, \zeta) = d_{\mathcal{F}}(ST\eta, T\eta) = d_{\mathcal{F}}(TS\eta, T\eta) < d_{\mathcal{F}}(SS\eta, S\eta) = d_{\mathcal{F}}(S\zeta, \zeta),$$

which is a contradiction. Thus, we obtain $S\zeta = \zeta$ and $T\zeta = TS\eta = ST\eta = S\zeta = \zeta$. Therefore, ζ is a common fixed point of S and T.

Suppose another common fixed point exists that is different from ζ, namely ζ'. Then by (10), we obtain

$$d_{\mathcal{F}}(\zeta, \zeta') = d_{\mathcal{F}}(S\zeta, S\zeta') < d_{\mathcal{F}}(T\zeta, T\zeta') = d_{\mathcal{F}}(\zeta, \zeta'),$$

which contradicts $\zeta \neq \zeta'$. Therefore, ζ is unique. □

In addition, the following example is given to support Theorem 3.7.

Example 3.8. Given $X = [0, 10]$ and $d_{\mathcal{F}} : X^2 \to \mathbb{R}$ is defined by

$$d_{\mathcal{F}}(\eta, \theta) = \begin{cases} (\eta - \theta)^2, \ \eta, \theta \in [0, 2]; \\ |\eta - \theta|, \ \text{otherwise.} \end{cases}$$

for any $(\eta, \theta) \in X^2$. Clearly, $d_{\mathcal{F}}$ satisfies (df$_1$) and (df$_2$). However, $d_{\mathcal{F}}$ does not satisfy triangle inequality since

$$3 = d_{\mathcal{F}}(0, 3) > d_{\mathcal{F}}\left(0, \frac{1}{2}\right) + d_{\mathcal{F}}\left(\frac{1}{2}, 3\right) = \frac{11}{4}.$$

Moreover, $d_{\mathcal{F}}$ satisfies (df$_3$) with $g(s) = \ln(s)$ for every $s \in (0, \infty)$ and $\alpha = \ln(2)$. Thus, $(X, d_{\mathcal{F}})$ is an $\mathcal{F}$-metric space. Next, defined d^* is metric as in Equation (1). Since X is a finite set, it is clear that (X, d^*) is a compact metric space. By Proposition 2.4, $(X, d_{\mathcal{F}})$ is an $\mathcal{F}$-compact $\mathcal{F}$-metric space. Let

$$S\eta = \begin{cases} \eta, \ \eta \in [0, 8]; \\ 8, \ \eta \in (8, 10] \end{cases}$$

and $T\eta = 5$ for every $\eta \in X$. Obviously, S is continuous mapping and $T \in C_S$. Suppose $\epsilon > 0$ is an arbitrary point, there exists $\delta > 0$ such that if we have $\epsilon \leq d_{\mathcal{F}}(S\eta, S\theta) < \epsilon + \delta$ implies $d_{\mathcal{F}}(T\eta, T\theta) = 0 < \epsilon$ for all $\eta, \theta \in X$. Thus, T is an (ϵ, δ)-S-contraction. Moreover, since the conditions of Theorem 3.7 hold, T and S have a unique common fixed point, i.e., $\eta^* = 5 = S(5) = T(5)$.

In the following, we establish a proposition regarding the metrizability result.

Proposition 3.9. Given $(X, d_{\mathcal{F}})$ is an $\mathcal{F}$-metric space with $(g, \alpha) \in \mathcal{F} \times [0, \infty)$, d^* is a metric as in (1) and S, T are a self-mapping on X with $T \in C_S$. If T is an (ϵ, δ)-S-contraction in the setting of $\mathcal{F}$-metric space, then T is an (ϵ, δ)-S-contraction as regards metric d^*.

Proof. Suppose $\eta, \theta \in X$ and $\epsilon > 0$ are arbitrary points and $\{s_n\}_{n=1}^q \subseteq X$ with $(s_1, s_q) = (\eta, \theta)$ and $q \geq 2$. Suppose $\{T(s_n)\}_{n=1}^q$ and $\{S(s_n)\}_{n=1}^q$ are the finite sequences in X. By (M1), there exists $\delta > 0$ such that

$$\epsilon \leq d_{\mathcal{F}}(S\eta, S\theta) < \epsilon + \delta \quad \text{implies} \quad d_{\mathcal{F}}(T\eta, T\theta) < \epsilon, \tag{13}$$

whenever $S\eta \neq S\theta$. By (1), we have

$$\left| \begin{array}{l} d^*(S(\eta), S(\theta)) = \inf\{\sum_{j=1}^{q-1} d_{\mathcal{F}}(S(s_j), S(s_{j+1})) : q \in \mathbb{N}, q \geq 2, (s_1, s_q) = (\eta, \theta)\}; \\ d^*(T(\eta), T(\theta)) = \inf\{\sum_{j=1}^{q-1} d_{\mathcal{F}}(T(s_j), T(s_{j+1})) : q \in \mathbb{N}, q \geq 2, (s_1, s_q) = (\eta, \theta)\}. \end{array} \right. \tag{14}$$

From (13) and (14), we obtain

$$\begin{aligned} &\inf\left\{\sum_{j=1}^{q-1} \epsilon : q \in \mathbb{N}, q \geq 2\right\} \leq d^*(S(\eta), S(\theta)) < \inf\left\{\sum_{j=1}^{q-1} \epsilon + \delta : q \in \mathbb{N}, q \geq 2\right\} \\ &\iff \epsilon \leq d^*(S(\eta), S(\theta)) < \epsilon + \delta, \end{aligned}$$

and

$$d^*(T(\eta), T(\theta)) < \inf\left\{\sum_{j=1}^{q-1} \epsilon : q \in \mathbb{N}, q \geq 2\right\} \iff d^*(T(\eta), T(\theta)) < \epsilon.$$

Therefore, T is an (ϵ, δ)-S-contraction as regards metric d^*. □

Corollary 3.10. *We conclude from Proposition 2.4 and Proposition 3.9 that Meir-Keeler contraction in $\mathcal{F}$-metric space is a direct result of Meir-Keeler contraction for usual metric space.*

For further research, we also establish a remark.

Remark 3.11. In Park and Bae (1981), the authors have proved a common fixed point theorem for Meir-Keeler contraction in complete metric space. In other words, they proved it in $\mathcal{F}$-complete $\mathcal{F}$-metric space with $f(s) = \ln(s)$ for all $s \in (0, \infty)$ and $\alpha = 0$. The triangle inequality property in metric has a significant role in the proof. Since $\mathcal{F}$-metric may not have triangle inequality property, the common fixed point theorem for Meir-Keeler contraction fails to prove in an $\mathcal{F}$-complete $\mathcal{F}$-metric space. An open question is whether the necessary conditions are sufficient for the theorem to be proven in $\mathcal{F}$-complete $\mathcal{F}$-metric space.

Acknowledgement

The authors delightedly acknowledge the Ministry of Higher Education Malaysia and Universiti Malaysia Terengganu for partially supporting this research under the Fundamental Research Grant Scheme (FRGS) Project Code FRGS/1/2021/STG06/UMT/02/1 and Vote no. 59659.

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Received: 17 August 2023
Accepted: 15 September 2023

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