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ON THE RELATION OF SEVEN CHAOS CHARACTERIZATIONS

(Hubungan antara Tujuh Ciri Kekalutan)

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ABSTRACT

In this work, we explore seven chaos-related notions in dynamical systems: locally everywhere onto, mixing, totally transitive, strong dense periodicity, blending, specification, and Devaney chaos. We analyze their interrelations, proving positive connections and providing counterexamples for negative ones. Our findings establish a hierarchy among these chaos characterizations, with the specification property at the top and blending, transitivity, and strong dense periodicity at the bottom in compact spaces. In shifts of finite type, these properties are equivalent, but this equivalence does not hold in shifts of infinite type.

Keywords: Devaney chaos; locally everywhere onto; totally transitivity; topologically mixing; strong dense periodic points; blending

ABSTRAK

Dalam kajian ini, kami meneroka tujuh konsep berkaitan kekalutan dalam sistem dinamik: sifat keseluruh setempat di mana-mana, pencampuran, transitif sepenuhnya, sifat berkalala tumpat yang kuat, pencampuran, spesifikasi, dan kekalutan Devaney. Kami menganalisis hubungan antara konsep-konsep ini dengan membuktikan hubungan positif dan memberikan contoh penyangkal untuk hubungan negatif. Penemuan kami menunjukkan hierarki antara ciri-ciri kekacauan ini, dengan sifat spesifikasi berada di puncak, manakala sifat pencampuran, transitiviti, dan sifat berkala tumpat yang kuat berada di kedudukan terendah dalam ruang padat. Dalam ruang anjakan jenis terhingga, sifat-sifat ini adalah setara, tetapi kesetaraan ini tidak berlaku dalam anjakan jenis tidak terhingga.

Kata kunci: kekalutan Devaney; keseluruh setempat di mana-mana; transitif sepenuhnya; pencampuran secara bertopologi; berkala tumpat yang kuat; pencampuran

1. Introduction

Beginning with the ingredients of Devaney chaos (transitivity, a dense set of periodic points, and sensitive dependence on initial conditions) (Devaney 2003), mathematicians have been developing the chaos theory of dynamical systems. Some researchers have focused on transitivity as a main ingredient of chaos. Consequently, by strengthening or weakening the condition of transitivity, many strong versions of transitivity were proposed, such as totally transitive, mixing, and locally everywhere onto (l.e.o.), with all these concepts being stronger than transitivity. Effah-Poku *et al.* (2018) considered systems with at least two of these properties are considered to be chaotic in a certain sence; bifurcation and period doubling, period three, transitivity and dense orbit, sensitive dependence to initial conditions, and expansivity. In 2022, Wong and Salleh (2022) studied the dynamical properties of set-valued dynamical systems; sensitivity, transitivity and mixing. They defined sensitivity on the set-valued dynamical systems and studied its properties. Recently Han *et al.* (2023) studied chaotic properties in memristive systems with details analysis of sensitivity and periodic points.

Other researchers have focused on the condition of dense periodic points. Dzul-Kifli and Good (2015) introduced the concept of the strong dense periodicity property (P_n dense for all n). Considering l.e.o. and P_n dense for all n as significant chaos characterizations, Baloush and Dzul-Kifli (2016) showed that l.e.o. implies chaos characterizations such as topologically mixing, totally transitive, and blending. On 1-step shifts of finite type over two symbols, l.e.o. and P_n dense for all n are equivalent (Baloush *et al.* 2016). Dzul-Kifli and Al-Muttairi (2015) have shown that for a shift of finite type over two symbols, the property of P_n dense for all n implies l.e.o. and totally transitive. However, the complete relations between the above-mentioned chaos notions, i.e., totally transitive, topologically mixing, locally everywhere onto, P_n dense for all n, Devaney chaos (DevC), (weakly) blending, and some other notions such as the specification property, have not been figured out until now. In this work, we investigate all relations between these chaos notions in compact spaces and shift spaces. We then provide hierarchy diagrams of these chaos characterizations in compact spaces and shifts of finite type (SFT). Finally, we highlight the differences in the relations between these chaos characterizations from SFT to shifts of infinite type (SIFT). Here are the definitions of the chaos properties:

Definition 1.1. A dynamical space (X, f) is said to be topologically transitive if for any nonempty open subsets $U, V \subset X$, there exists $n > 0$ such that $f^n(U) \cap V \neq \emptyset$ (Devaney 2003).

Definition 1.2. A dynamical space $(X, f)$is said to be totally transitive if f^n is transitive for all $n \geq 1$ (Sabbaghan & Damerchiloo 2011).

Definition 1.3. A function $f: X \to X$ is said to be topologically mixing if for any nonempty open sets $U, V \subset X$, there exists an $N \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$ for all $n > N$ (Denker *et al.* 1976).

Definition 1.4. A dynamical space (X, f) is said to be locally everywhere onto (l.e.o. for short) if for every open set $U \subseteq X$ there exists a positive integer n such that $f^n(U) = X$ (Good *et al.* 2006).

Definition 1.5. A dynamical space (X, f) is said to be (weakly) blending if for any pair of nonempty open sets U and V in X, there exists an $n > 0$ such that $f^n(U) \cap f^n(V) \neq \emptyset$, and strongly blending if for any pair of nonempty open sets U and V in X, there is some $n > 0$ such that $f^n(U) \cap f^n(V)$ contains an open set (Crannell 1995).

Definition 1.6. (Baloush *et al.* 2016) A dynamical space (X, f), has the strong dense periodicity property if the set of periodic points P_n is dense in X, where

$$P_n = \{x \in X : x \text{ is a periodic point of prime period } k \text{ for some } k \geq n\}.$$

For convenience, we may also refer to this property as having P_n dense for *all* $n \in \mathbb{N}$.

Definition 1.7. A dynamical system (X, f) has the specification property (briefly, *Sp*) if for any $\epsilon > 0$ there exists an integer M_ϵ such that for any $k \geq 2$, for any k points $x_1, \dots, x_k \in X$, for any integers $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$ with $a_i - b_{i-1} \geq M_\epsilon$ for $2 \leq i \leq k$ and for any integer p with $p \geq M_\epsilon + b_k - a_1$, there exists a point $x \in X$ with $f^p(x) = x$ such that $d(f^n(x), f^n(x_i)) \leq \epsilon$ for $a_i \leq n \leq b_i, 1 \leq i \leq k$ (Bowen 1971).

2. Implication Relation between Chaos Characterizations on Compact Spaces

In this section, we present the complete relationships among the seven chaos characterizations mentioned above for the dynamical system (X, f), where f is a continuous function on a compact space X. We found that the relations among weakly blending, strong dense periodicity property, totally transitive, Devaney chaos, topologically mixing, locally everywhere onto, and the specification property form a hierarchy based on their ability to imply other chaos characterizations, as shown in Figure 1.

Figure 1: Hierarchy of chaos characterizations based on their ability to imply others

The specification property is at the top of the hierarchy since this chaos characterization implies the other five chaos properties, i.e., topologically mixing, DevC, weakly blending, totally transitive, and a strong dense periodicity property. The implication relations are represented by the white arrows shown in the hierarchy diagram. However, the specification property does not imply locally everywhere onto, and therefore no arrow from specification property to locally everywhere onto appears in the hierarchy. Theorems and counterexamples are provided in this section to support these implication relations. As we look further into the hierarchy, there are five other levels, as shown and explained in Figure 1. Locally everywhere onto, topologically mixing, and Devaney chaos belong to the second, third, and fourth levels of the hierarchy, respectively, and the chaos characterizations implied or not implied by these characterizations are indicated by the existence of white arrows. Weakly blending, totally transitive, and a strong dense periodicity property are at the lowest level of the hierarchy because they do not imply any other chaos characterization.

The first chaos characterization to look at is l.e.o.

Theorem 2.1. *$l.e.o$ implies topologically mixing, totally transitive but does not imply $DevC$ nor P_n dense for all n (Baloush & Dzul-Kifli 2017).*

Theorem 2.2. *$l.e.o$ implies weakly blending.*

Proof. To show that $l.e.o$ implies weakly blending, let U, V be any two nonempty open sets in X. Since f is $l.e.o$, then there exists $n_1, n_2 > 0$ such that

$$f^{n_1}(U) = f^{n_2}(V) = X.$$

Without loss of generality, let $n_1 > n_2$. Then

$$f^{n_1}(U) = f^{n_1}(V) = X.$$

So

$$f^{n_1}(U) \cap f^{n_1}(V) = X.$$

Since X is an open set, and since $f^{n_1}(U) \cap f^{n_1}(V) \neq \emptyset$ for some $n > 0$, then f is weakly blending. □

Example 2.3. Let $n \in \mathbb{N}$ and Z_{n+1} be a cyclic group with $n+1$ elements. Let $X_n = (\mathbb{Z}_{n+1})^{\infty} = \{(x_m)_{m=1}^{\infty} : x_m \in \mathbb{Z}_{n+1}, m \in \mathbb{N}\}$ be the product topological space of countably infinite copies of Z_{n+1}, where Z_{n+1} endowed with the discrete topology. It is well known that X_n is an compact, perfect and has countable base containing clopen sets. We can choose this base to be consist of cylinder sets, i.e.,

$$[z_1, \cdots, z_k] = \{(x_m)_{m=1}^{\infty} \in X_n : x_1 = z_1, \cdots, x_k = z_k\}$$

where $k \in \mathbb{N}$ and $z_1, \cdots, z_k$ is an arbitrary sequence of elements of $\mathbb{Z}_{n+1}$ of length k. Define the map $f_n : X_n \to X_n$, by $f_n((x_m)_{m=1}^{\infty}) = (y_m)_{m=1}^{\infty}$, where

$$y_m = \begin{cases} x_{m+1} & \text{if } x_1 \neq x_{n+1} \\ 1 + x_{m+1} & \text{if } x_1 = x_{n+1} \end{cases}$$

for all $m \in \mathbb{N}$.

Lemma 2.4. *(Guirao et al. 2009) Let $n \in \mathbb{N}$, and f_n be the function defined in Example 2.3. Then*
(1) f_n is continuous,
(2) f_n has no periodic points with prime period n,
(3) if $n \geq 3$, then f_n has the Sp,
(4) f_n is $l.e.o$.

Theorem 2.5. *$l.e.o$ does not imply the Sp.*

Proof. Consider the function f_n^n where f_n is defined in Example 2.3. By Lemma 2.4, since f_n is $l.e.o$, so does f_n^n. Since f_n has no periodic points with prime period n, so f_n^n has no fixed points. By Sharkovsky's, f_n^n does not have any periodic point. Hence f_n^n is not $DevC$ nor has P_n dense for all n. Also since f_n^n has no periodic points, then it does not satisfy the definition of the Sp, hence f_n^n dose not have the Sp. □

Next, we move to the weakly blending property and the following are counterexamples to show why weakly blending does not imply any other chaos characterizations.

Example 2.6. Consider the function

$$f(x) = \begin{cases} -2x-2 & if \quad 1 \le x \le -\frac{1}{2} \\ 2x & if \quad |x| < \frac{1}{2} \\ 2-2x & if \quad \frac{1}{2} \le x \le 1 \end{cases}$$

which defined on the interval $[-1, 1]$ (Crannell 1995).

Lemma 2.7. *f is weakly blending but not topologically transitive.*

Proof. The map f is weakly blending since every open subinterval of $[-1,1]$ eventually maps onto an interval which contains the fixed point of f which is located at the origin. To prove that f is not topologically transitive, let $U = (0, 1)$ and $V = (-1,0)$, then for any positive integer $n, f^n(U) = f^n(0,1) = (0,1)$. Hence, $f^n(U) \cap V = \emptyset$, for all $n > 0$. Therefore f is not topologically transitive. ☐

Theorem 2.8. *Weakly blending does not imply the $Sp, l.e.o$, topologically mixing, Devaney chaos and totally transitive.*

Proof. We will consider the map f as defined in Example 2.6. By Lemma 2.7, f is weakly blending but not topologically transitive. Since f is not topologically transitive, then it is not $DevC, l.e.o$, topologically mixing, totally transitive, nor has the Sp because transitivity is weaker then $l.e.o$, topologically mixing, totally transitive, and the Sp. Therefore, weakly blending does not imply $l.e.o$, topologically mixing, totally transitive, $DevC$ and neither the Sp. ☐

Theorem 2.9. *Weakly blending does not imply P_n dense for all n.*

Proof. Consider the map f_n^n where f_n is defined in Example 2.3. By Theorem 2.5, we proved that f_n^n is $l.e.o$ but does not have P_n dense for all n. Since f_n^n is $l.e.o$, then it is weakly blending by Theorem 2.2. Hence f_n^n is weakly blending but does not have P_n dense for all n. Therefore, weakly blending does not imply P_n dense for all n. ☐

Next, we move to the property of totally transitive, as follows.

Theorem 2.10. *Totally transitive does not imply $l.e.o$, topologically mixing, $DevC$ nor P_n dense for all n (Baloush & Dzul-kifli 2017).*

Example 2.11. The irrational rotation $R_\alpha: S_1 \to S_1$ is defined on unit circle by $R_\alpha: \theta \to \theta + \alpha \pmod{2\pi}$ where $\theta \in S_1$, and $\frac{\alpha}{2\pi} \in \mathbb{R}\backslash\mathbb{Q}$.

Theorem 2.12. *Totally transitive does not imply weakly blending.*

Proof. To prove this theorem, we will consider the irrational rotation $R_\alpha: S_1 \to S_1$ as defined in Example 2.11 and show that R_α is totally transitive but not weakly blending. Let $\theta \in S_1$, since the orbit of $\theta, O^+(\theta)$ is a sequence bounded by 0 and 2π, so it must have a convergent subsequence. Hence, for $\epsilon > 0$, there exist positive integers r and s with

$$|R_\alpha^s(\alpha) - R_\alpha^r(\alpha)| < \epsilon.$$

If we let $m = r - s$, where $r > s$, then since R_α preserves arc lengths, we have

$$|R_\alpha^m(\theta) - \theta| = \left|R_\alpha^s\left(R_\alpha^m(\theta)\right) - R_\alpha^s(\theta)\right| = |R_\alpha^s(\alpha) - R_\alpha^r(\alpha)| < \epsilon.$$

Let us now see the behaviour of the arc $\left(\theta, R_\alpha^m(\theta)\right)$ under $R_\alpha^m(\theta)$,

$$|R_\alpha^m(R_\alpha^m(\theta) - \theta)| = |R_\alpha^{2m}(\theta) - R_\alpha^m(\theta)| < \epsilon$$

and

$$|R_\alpha^m(R_\alpha^2(\theta) - R_\alpha^m(\theta))| = |R_\alpha^{3m}(\theta) - R_\alpha^{2m}(\theta)| < \epsilon$$

and so on. So

$$O^+(\theta) = \theta,\ R_\alpha^m(\theta),\ R_\alpha^{2m}(\theta),\ R_\alpha^{3m}(\theta), \cdots,$$

which partitions the circle S^1 into arcs of length less than ϵ. Since we let ϵ be arbitrary small, we can choose any open arc V such that ϵ is less than the length of V. Then $O^+(\theta)$ intersects V and so R_α is transitive. Since $R_\alpha^m = R_{m\alpha}$ for any integer $m > 0$, and $m\alpha$ is also irrational relative to 2π, then R_α is totally transitive. To show that R_α is not weakly blending, let U and V be any two open sets such that $U \cap V = \emptyset$. Observe that the irrational rotation R_α is an isometry, so it preserves lengths, i.e., $d(x, y) = d\left(R_\alpha(x), R_\alpha(y)\right)$. So if for every $x \in U$ and for every $y \in V, d(x, y) = \epsilon$, then for any $n > 0$, we have $d\left(R_\alpha^n(x), R_\alpha^n(y)\right) = \epsilon$, which means that, points that are close together stay close together under a rotation map. Hence for every $n > 0$, we obtain $R_\alpha^n(U) \cap R_\alpha^n(V) = \emptyset$. Therefore, R_α is not weakly blending. ☐

Theorem 2.13. *Totally transitive does not imply the Sp.*

Proof. Consider the function f_n^n where f_n is defined in Example 2.3. By Theorem 2.5, we proved that f_n^n is $l.e.o$ but does not have the Sp, and then by Theorem 2.1, we conclude that totally transitive does not imply the Sp. ☐

The next chaos characterization to look at is the strong dense periodicity property i.e., whenever the set

$$P_n = \{x \in X : x \text{ is a periodic point of prime period } k \text{ for some } k \geq n\}.$$

is dense for all n.

Theorem 2.14. *P_n is dense for all n does not imply $l.e.o, DevC$, totally transitive, neither topologically mixing (Baloush & Dzul-Kifli 2017).*

Example 2.15. Let $f: S^1 \times [0,1] \to S^1 \times [0,1]$ be defined as $f(\theta, a) = (\theta + 2\pi a, a)$.

Theorem 2.16. *P_n is dense for all n does not imply weakly blending.*

Proof. To proof this theorem, we will consider the function $f: S^1 \times [0,1] \to S^1 \times [0,1]$ as defined in Example 2.15 and show f that has P_n dense for all n, but is not weakly blending. To show that P_n is dense for all n, we must prove that for every n, every open subset $A \times B \subset S^1 \times [0,1]$ contains a periodic point of prime period greater or equal to n. We claim that for every n there exists an integer $q \geq n$ such that $\frac{p}{q} \in B$ for some integer p (p is neither q nor 1). Let $q > 2$ be large enough such that when we partition [0,1] into q sub-intervals with the same length, then there exists endpoint (except 0 and 1) of the sub-interval which lies in B of the form $\frac{p}{q}$ where $p = 1, 2, \dots, q - 1$. Therefore, for every n, there exists $q \geq n$ and $q \neq 2$ such that $\left(\theta, \frac{p}{q}\right) \in A \times B$ for any $\theta \in A$ and $p = 1, 2, \dots, q - 1$. Then

$$\begin{aligned} f^q\left(\theta, \frac{p}{q}\right) &= \left(\theta + q\left(2\pi \frac{p}{q}\right), \frac{p}{q}\right) \\ &= \left(\theta + 2\pi q, \frac{p}{q}\right) \\ &= \left(\theta, \frac{p}{q}\right) \end{aligned}$$

where q is a prime period for $\left(\theta, \frac{p}{q}\right)$. If not then there exists $k < q$ such that $\frac{2k\pi p}{q}$ is an integer, i.e., either $q = 2, q = k$ or $q = p$, which is a contradiction. Hence, $\left(\theta, \frac{p}{q}\right)$ is a periodic point of prime period q and then P_n is dense for every n. Now f is a rotation by an angle of 2π on the unit circle at axis $-a, S^1 \times \{a\}$. So, f does not behave as a chaotic system since every point only moves regularly within its own circle at the same axis. Since f fixes $S^1 \times \{a\}$ for all $a \in [0,1]$, take U and V such that

$$U = S^1 \times \left(0, \frac{1}{4}\right) \, and \, V = S^1 \times \left(\frac{1}{2}, 1\right).$$

Then $f^n(U) \cap f^n(V) = \phi$, for all $n > 0$. So, this system is not weakly blending. ☐

Example 2.17. (Dzul-Kifli & Good 2015) Let $\mathrm{D} = \{\mathrm{re^{i\theta}} \in \mathbb{C} : \mathrm{r} \in [0,1], \theta \in [0, 2\pi)\}$ be the closed unit disk in the complex plane. Define $\mathrm{f}: \mathrm{D} \to \mathrm{D}$ by $\mathrm{f}(\mathrm{re^{i\theta}}) = \mathrm{re^{i(\theta + 2r\pi)}}$. Then f is a homeomorphism of D and the restriction $\mathrm{f_r}$ of f to $\mathrm{C_r} = \{\mathrm{z} : |\mathrm{z}| = \mathrm{r}\}$ is a rotation of order r.

Lemma 2.18. *Let f and D are as defined in Example 2.17, then (f, D) has P_n is dense for all n but not transitive (Dzul-Kifli & Good 2015).*

Theorem 2.19. *P_n is dense for all n does not imply the Sp.*

Proof. We will consider $f: D \to D$ as defined in Example 2.17. By Lemma 2.18, f has P_n is dense for all n and since f is not transitive, then it does not have the Sp because transitivity is weaker than the Sp. Therefore f is the counterexample to prove this theorem. ☐

Now we look at the Devaney chaos and consider the following example.

Example 2.20. Let f: [0,1] → [0,1] be defined by

$$f(x) = \begin{cases} \frac{1}{2} + 2x & if \quad 0 \le x \le \frac{1}{4} \\ \frac{3}{2} - 2x & if \quad \frac{1}{4} \le \ x \le \frac{1}{2} \\ 1 - x & if \quad \frac{1}{2} \le x \le 1 \end{cases}$$

Lemma 2.21. *Let $f: [0,1] \to [0,1]$ be defined as in Example 2.20, then f is topologically transitive but not totally transitive neither weakly blending.*

Proof. Banks in 1997showed that f is topologically transitive. To prove that f is not totally transitive, let $U = \left(0, \frac{1}{2}\right)$, and $V = \left(\frac{1}{2}, 1\right)$. Then $f(U) = f\left(\left(0, \frac{1}{2}\right)\right) = \left(\frac{1}{2}, 1\right) = V$ and $f(V) = f\left(\left(\frac{1}{2}, 1\right)\right) = \left(0, \frac{1}{2}\right) = U$. i.e., U and V are are invariant under f^2. So f^2 is not topologically transitive on [0,1]. Therefore f is not totally transitive. Finally, f is not weakly blending, for that, let $U = \left(0, \frac{1}{2}\right)$, and $V = \left(\frac{1}{2}, 1\right)$, and let $n > 0$ be an integer. If n is odd, then $f^n(U) = V$, $f^n(V) = U$, and if n is even, then$f^n(U) = U$, $f^n(V) = V$. Hence, for any case of n we have $f^n(U) \cap f^n(V) = \emptyset$. Therefore, f is not weakly blending. ☐

Theorem 2.22. *$DevC$ does not imply totally transitive, topologically mixing, $l.e.o$, weakly blending neither the Sp.*

Proof. Let us consider $f: [0,1] \to [0,1]$ as defined in Example 2.20. By Lemma 2.21, f is topologically transitive but not totally transitive neither weakly blending. Since f is a transitive interval map, then it is $DevC$ because transitivity is equivalent to $DevC$ on interval (Bowen 1971). Also, since f is not totally transitive, then it is not $l.e.o$ neither topologically mixing since $l.e.o$ and topologically mixing are stronger than totally transitive. Finally, since f is not topologically mixing, then it does not have the Sp since the Sp is stronger than topologically mixing (Alsedà *et al*. 2003). Therefore, $DevC$ does not imply totally transitive, topologically mixing, $l.e.o$, weakly blending neither the Sp. ☐

Lemma 2.23. *Let X be a compact metric space without isolated points. If $f: X \to X$ is a continuous $DevC$ function, then it has P_n dense for all n (Dzul-Kifli & Good 2015).*

Lemma 2.24. *If a dynamical system (X, f) is $SDIC$, then it has no isolated points.*

Proof. Let X has $SDIC$. Suppose $x \in X$ is an isolated point, i.e. the singleton $U = \{x\}$ is open in X, then there is no two distinct points $y, z \in U$ such that for $n > 0$, $d(f^n(y), (f^{n(z)}) > c$, for some $c > 0$, which is a contradiction since X is $SDIC$. Therefore X has no isolated point.☐

Theorem 2.25. *$DevC$ implies P_n is dense for all n.*

Proof. Let (X, f) be a $DevC$ dynamical system. Then it is $SDIC$, and hence has no isolated points, by Lemma 2.24. Hence (X, f) by Lemma 2.23, has P_n is dense for all n. Therefore $DevC$ implies P_n is dense for all n. ☐

We move to the next chaos property, the topologically mixing.

Theorem 2.26. *Topologically mixing implies totally transitive but does not imply $DevC$ nor P_n is dense for all n (Baloush & Dzul-Kifli 2017).*

Lemma 2.27. *For any continuous map f, it is weakly mixing iff for any non-empty open sets U and V there is a $k \geq 1$ such that $f^k(U) \cap V \neq \phi$ and $f^k(V) \cap V \neq \phi$ (Banks 2005).*

Theorem 2.28 *Topologically mixing implies weakly blending.*

Proof. Let U and V be any two nonempty open sets in X. Since topologically mixing implies weakly mixing, then by Lemma 2.27 we have,

$$f^k(U) \cap V \neq \emptyset \text{ and } f^k(V) \cap V \neq \phi, \text{ for some } k \geq 1.$$

Now

$$(f^k(U) \cap V) \cap (f^k(V) \cap V) = (f^k(U) \cap f^k(V)) \cap V.$$

We have two cases, as follows;

- Case 1: $(f^k(U) \cap f^k(V)) \cap V \neq \emptyset$. Since $V \neq \emptyset$, then $f^k(U) \cap f^k(V) \neq \emptyset$ and we are done.
- Case 2: $(f^k(U) \cap f^k(V)) \cap V = \emptyset$. Since V is a nonempty open set, then for all $x \in V$, $x \notin f^k(U) \cap f^k(V)$. Hence, for all $x \in V$, $x \notin f^k(U)$ or $x \notin f^k(V)$, which implies that

$$f^k(U) \cap V = \emptyset \text{ or } f^k(V) \cap V = \emptyset,$$

which is a contradiction since f is weakly mixing. Hence f is weakly blending. ☐

Lemma 2.29. *Let $f: I \to I$ be a λ-expanding interval map with $\lambda > N$, where N is a positive integer. Then, for every nondegenerate sub-interval J, there exists an integer $n \geq 0$ such that $f^n(J)$ contains at least N distinct critical points (Ruette 2016).*

Lemma 2.30. *Let $f: [a, b] \to [a, b]$ be a topologically mixing interval map. Then f is $l.e.o$ if and only if both a and b are accessible (Hasselblatt & Katok 2003).*

Example 2.31. *(Ruette 2016) Let $(a_n)_{n\in\mathbb{Z}}$ be a sequence of points in $(0,1)$ such that $a_n < a_{n+1}$ for all $n \in \mathbb{Z}$, and*

$$\lim_{n\to-\infty} a_n = 0 \text{ and } \lim_{n\to+\infty} a_n = 1.$$

For all $n \in \mathbb{Z}$, we let $I_n = [a_n, a_{n+1}]$ and $f_n: I_n \to I_{n-1} \cup I_n \cup I_{n+1}$ is given by $f_n(a_n) = a_n$, $f_n(a_{n+1}) = a_{n+1}$, $f_n\left(\frac{2a_n+a_{n+1}}{3}\right) = a_{n+2}$, $f_n\left(\frac{a_n+2a_{n+1}}{3}\right) = a_{n-1}$.

Then we define the map $f: [0,1] \to [0,1]$ by $f(0) = 0$, $f(1) = 1$ and for all integer n, for all $x \in I_n$, $f(x) = f_n(x)$.

Lemma 2.32. *Let $f: [a,b] \to [a,b]$ be defined as in Example 2.31. Then f is topologically mixing but not $l.e.o$.*

Proof. It is obvious that the end points 0 and 1 are not accessible because for any $n \geq 1$ and any $x \in (0,1), f^n(x) \neq 0$ and similar for the end point 1. Hence, f is not $l.e.o$ by Lemma 2.30. Now we will show that f is topologically mixing (Bowen 1971), let J be a non-degenerate subinterval of [0,1]. Since f sends the middle third of I_n (specifically $\left(\frac{2a_n+a_{n+1}}{3}, \frac{a_n+2a_{n+1}}{3}\right)$) into an adjacent interval, the map f stretches the interval in such a way that the length of the image under f is three times larger than the original interval, then f is 3 −expanding (Ruette 2016). Hence, by Lemma 2.29, there exists $n \geq 0$ such that $f^n(J)$ contains two distinct critical points. This implies that $f^{n+1}(J)$ contains I_k for some $k \in \mathbb{Z}$. Now for all $k \in \mathbb{Z}$ and for all $m \geq 0$, we have

$$[a_{k-m}, a_{k+m+1}] \subset f^m(I_k).$$

Since, for a given $k \in \mathbb{Z}$, the lengths of $[0, a_{k-m}]$ and $[a_{k+m+1}, 1] \to 0$ when $m \to \infty$, then for all $\epsilon > 0$, there exists M such that $L = [\epsilon, 1-\epsilon] \subset f^m(J)$, for all $m \geq M$. Hence, $f^m(J) \cap L \neq \emptyset$, for all $m \geq M$. Therefore f is topologically mixing. ☐

Theorem 2.33. *Topologically mixing does not imply $l.e.o$.*

Proof. Consider $f: [a,b] \to [a,b]$ as defined in Example 2.31. By Lemma 2.32, f is topologically mixing but not $l.e.o$. Therefore, topologically mixing does not imply $l.e.o$. ☐

Theorem 2.34. *Topologically mixing does not imply the Sp.*

Proof. In Theorem 2.5 we proved that the map f_n^n where f_n is as defined in Example 2.3 is $l.e.o$ and does not have the Sp. Since by Theorem 2.1, every $l.e.o$ is topologically mixing, then f_n^n is a topologically mixing but does not have the Sp. Therefore, on compact space topologically mixing does not imply the Sp. ☐

Theorem 2.34. *The Sp implies topologically mixing, and then totally transitive (Alsedà et al. 2003).*

Theorem 2.35. *The Sp implies weakly blending.*

Proof. By Theorem 2.35, the Sp implies topologically mixing, and since every topologically mixing map by Theorem 2.28 is weakly blending, the proof is complete. ☐

Theorem 2.36. *If (X, f) satisfies the Sp, then f has dense periodic points and f is topologically mixing (Denker et al. 1976).*

Lemma 2.37. *A topologically mixing map (on a space with more than one point) has $SDIC$ (Hasselblatt & Katok 2003).*

Lemma 2.38. *Let X be a compact metric space without isolated points. If $f: X \to X$ is a continuous $DevC$ function, then it has P_n is dense for all n (Dzul-Kifli & Good 2015).*

Theorem 2.39. *The Sp implies P_n is dense for all n.*

Proof. Since $f: X \to X$ has the Sp, then f is transitive, topologically mixing and has dense periodic points by Lemma 2.37. Also, since f is topologically mixing then it is $SDIC$ by Lemma 2.38, and hence the space X has no isolated points by Lemma 2.24. So, the dynamical system (X, f) is a $DevC$ compact metric space without isolated points. Hence by Theorem 2.39 the system (X, f) has P_n is dense for all n. Therefore, the Sp implies P_n is dense for all n. ☐

Theorem 2.40. *The Sp implies $DevC$.*

Proof. By Lemma 2.37, the Sp implies topologically mixing and dense periodic points. Now every topologically mixing map is topologically transitive and by Lemma 2.38, topologically mixing implies $SDIC$. Therefore, the Sp implies $DevC$. ☐

Lemma 2.41. *A topologically mixing interval map $f: I \to I$ has the Sp (Ruette 2016).*

Theorem 2.42. *The Sp does not imply $l.e.o.$*

Proof. Consider the interval map $f : [0,1] \to [0,1]$ as in Example 2.31. By Lemma 2.32, f is a topologically mixing interval map but not $l.e.o.$ Since f is topologically mixing interval map, then by Lemma 2.41 it has the Sp. Hence, f satisfied the Sp but is not $l.e.o.$ Therefore, the Sp does not imply $l.e.o.$ ☐

3. Implication Relation between Chaos Characterizations on Shift of finite type

In this part, we look at a more specific space, the shift space. There are two types of shift space, i.e., shift of finite type (SFT) and shift of infinite type (SIFT), which are dependent on the number of forbidden blocks used to define the space. On SFT, we found that the hierarchy of the seven chaos characterizations consists of only two levels, as presented in Figure 2. There are five equivalent chaos characterizations located in the top level of the hierarchy. In the shift of infinite type, the equivalences are not true, and some counterexamples are provided in the following section.

A shift space X is a collection of all sequences over n symbols in alphabet set $\mathcal{A} = \{1,2,\cdots,n-1\}$ such that there are some blocks that are not allowed to appear in the sequences. Therefore, $X \subseteq \mathcal{A}^{\mathbb{N}}$. Whenever the set of forbidden blocks is empty, the space is called full-n-shift, Σ_n. If the number of forbidden blocks is finite, then it is called SFT, and vice versa. The shift map σ is a map that shifting a sequence in X to the left, i.e $\sigma(s_0 s_1 s_2 \cdots) = s_1 s_2 \cdots$ for $s_0 s_1 s_2 \cdots \in X$. The space is a metric space which is equipped with metric

$$d(\mathrm{s},\mathrm{t}) = \begin{cases} 0 & if \quad s = t \\ 2^{-j} & if \quad s \neq t \end{cases}$$

where $j \in \mathbb{N}$ is the smallest number such that $s_j \neq t_j$. Therefore, the topology on any shift space X is the topology induced by the metric d . Note that for any open ball U in shift space X there exists an allowed block w in X of length l such that $U = \{\mathrm{t} \in X : t_0 t_1, \dots, t_{l-1} = w\}$. We denote U as U_w to indicate that U is generated by the block w. The shift map σ is continuous on this topological space.

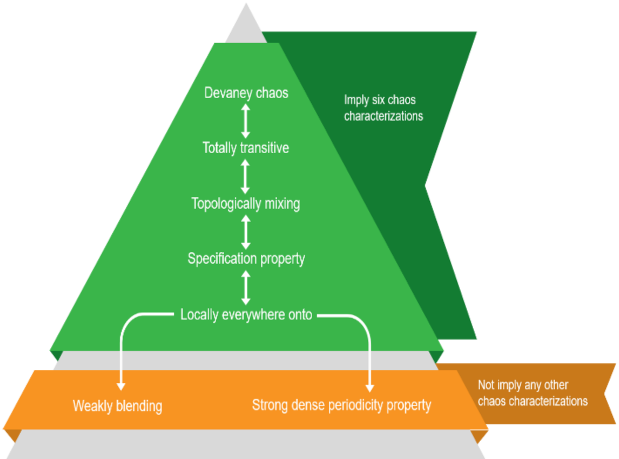

Figure 2: Hierarchy of chaos characterizations based on their ability to imply others on SFT

To show the equivalence of five chaos characterizations on a shift of finite type, let us recall some implications found earlier, as follows:

Proposition 3.1. *On SFT, totally transitive implies topologically mixing and* $l.e.o$ *(Crannell 1998).*

Proposition 3.2. *On SFT, topologically mixing implies the Sp (Denker et al. 1976).*

Proposition 3.3. *On SFT, transitive implies Devaney chaos (Dzul-Kifli 2011).*

Theorem 3.4 *On SFT, the following chaos properties are equivalent:*
(1) Totally transitive
(2) Topologically mixing
(3) Sp.
(4) $l.e.o.$
(5) $DevC.$

Proof. By Proposition 3.1, totally transitive implies topologically mixing, and by Proposition 3.2, topologically mixing implies the specification property. By Theorem 2.26 and Theorem 2.35, specification property implies totally transitive on compact metric spaces. By Proposition 3.1, totally transitive implies $l.e.o$ on SFT. Therefore, specification property implies $l.e.o$ on SFT. On compact metric space, $l.e.o$ implies totally transitive by Theorem 2.1, so does on SFT. To show that $DevC$ implies $l.e.o$, let $X \subseteq \Sigma_k$ be Devaney chaotic shift of finite type and we aim to show that X is $l.e.o$. For this we use the induction method on k. Let us assume that it is true for Σ_k, we need to show that a Devaney chaotic SFT $Y \subset \Sigma_{k+1}$ is $l.e.o$. Let

$$\mathcal{A} = \{0,1,\cdots,k\}, \text{and } \mathcal{A}' = \{0,1,\cdots,k-1\}$$

We define $Y' \subset \Sigma_k$ as a shift space over the alphabet $\mathcal{A}'$, where

$Y' = \{\mathbf{x}' = (x'_i)_{i \in \mathbb{N}}: x'_i \in \mathcal{A}' \text{for all } i \in \mathbb{N} \mid \exists \mathbf{x} \in Y: if\ x_i = k, \text{then } x'_j = 0 \text{ and}$
$if\ x_i \neq k, \text{then } x'_j = x_i\}$

Therefore, for every block w' in Y', there exists a block w in Y such that we get w' by replacing $k's$ in W with $0'$. We claim that Y' is a Devaney chaotic shift of finite type over k alphabets. Y' is a subshift of finite type since every forbidden block $\mathcal{F}'$ in Y' is also a forbidden block $\mathcal{F}$ in Y, that means $\mathcal{F}' \subseteq \mathcal{F}$ and therefore Y' has a finite number of forbidden blocks.

To show that Y' is Devaney chaotic, let u', v' be two allowed blocks in Y', and let u, v be allowed blocks in Y with respect to u' and v', respectively. Now we must show that for every u', v' in Y' there exists a block w' in Y'such that $u'w'v' \in Y'$. Since Y is transitive then for every two allowed blocks u, v in Y there exists a block w in Y such that $u\ w\ v \in Y$. By replacing u, v, w with u', v' and w', respectively, then we have $u'w'v' \in Y'$. Hence, $Y'$is transitive. Therefore, by Proposition 3.3 Y' is Devaney chaotic and then by our assumption, Y' has $l.e.o$. Let U' be a nonempty open set in Y', where U' is generated by w', and let w be an allowed block in Y with respect to w', and U is a nonempty open set in Y generated by w, i.e., $U = Y_w$.

Now we will show that Y is $l.e.o$, i.e., there exists a positive integer n such that $\sigma^n(U) = Y$, i.e.

$$\forall y \in Y, \exists u \in U \text{ such that } \sigma^n(u) = y.$$

Let $y \in Y$, then by the definition of Y'we get y' by replacing $k's$ in y by $0's$. Since Y' is $l.e.o$ then there exists a positive integer n such that $\sigma^n(U') = Y'$, i.e.,

$$\forall y' \in Y', \exists u' \in U' \text{ such that } \sigma^n(u') = y' .$$

Let u be an allowed block in Y such that we produce u' by replacing $k's$ in U with $0's$, then $|u| = |u'|$. Also, let $y \in Y$ such that we produce y' by replacing $k's$ in y with $0's$. Then $u \in U$ and$y \in Y$ and $\sigma^n(u) = y$. Therefore, $\sigma^n(Y) = U$, and hence Y is $l.e.o$. Finally, by Theorem 3.4, $l.e.o$ implies specification property and by Theorem 2.41, on compact metric space, specification property implies Devaney chaos, so does on SFT. ☐

However, the **s**trong dense periodicity property is weaker than these five chaos characterizations.

Theorem 3.5. *On SFT $l.e.o$ implies P_n dense for all n (Baloush & Dzul-Kifli 2018).*

The converse of the above theorem is not true by the following counterexample.

Example 3.6. Let $A = \{1, 2, 3, 4\}$ be an alphabet. Let $X = X_{\mathcal{F}}$ be a SFT over A with a set of forbidden blocks $\mathcal{F} = \{11, 13, 14, 23, 24, 31, 32, 33, 41, 42\}$.

Lemma 3.7. *Let $X = X_{\mathcal{F}}$ be a SFT as defined in Example 3.6. Then X has the P_n dense for all n but is not transitive neither weakly blending.*

Proof. Let $n \in \mathbb{N}$ and X_u be an open set in $X_{\mathcal{F}}$. Let $X_{\mathcal{F}_1}$ be a SFT with $\mathcal{F}_1 = \{11\}$ and $\forall\, \mathbf{x} \in X_{\mathcal{F}_1}$, $x_i \neq$ 3,4 for every $i \in \mathbb{N}$. Also let $X_{\mathcal{F}_2}$be a SFT with $\mathcal{F}_2 = \{33\}$ and $\forall\, \mathbf{x} \in X_{\mathcal{F}_2}$, $x_i \neq$ 1,2 for every $i \in \mathbb{N}$. Then

$$X_{\mathcal{F}} = X_{\mathcal{F}_1} \cup X_{\mathcal{F}_2}.$$

Since $\mathcal{F}_1 = \{11\}$ and 11 is the only forbidden block in $X_{\mathcal{F}_1}$, then $X_{\mathcal{F}_1}$ is topologically conjugate to the golden mean shift X_{GMS} ($X_{\mathcal{F}_1} \cong X_{GMS}$) which has P_n dense for all n. Similarly, since $\mathcal{F}_2 = \{33\}$ and 33 is the only forbidden block in $X_{\mathcal{F}_2}$, then $X_{\mathcal{F}_2}$ is conjugate to the golden mean shift X_{GMS} ($X_{\mathcal{F}_2} \cong X_{GMS}$). Now for the open set X_u we have either $X_u \subseteq X_{\mathcal{F}_1}$ or $X_u \subseteq X_{\mathcal{F}_2}$. Hence, in both cases, for every open set X_u and for every $n \in \mathbb{N}$ there exists a periodic point **x** of prime period k for some $k \geq n$ such that x $\in X_u$. Therefore, $X_{\mathcal{F}}$ has the property of P_n dense for all n.

To show that X is not transitive, let us consider the blocks $u = 1$ and $v = 3$. Then u, v are two allowed blocks in X but there is no allowable block w in X such that uwv is allowed. Therefore, X is not transitive. Finally, X is not weakly blending, for that, let $U = X_{12}$ and $V = X_{43}$ be two open sets in X. Then for every $n \in \mathbb{N}$ we have $f^n(U) \cap f^n(V) = \emptyset$. Therefore, X is not weakly blending. ☐

Theorem 3.8. *On SFT, the property of P_n dense for all n does not imply $l.e.o$ neither weakly blending.*

Proof. Consider the shift of finite type X as defined in Example 3.6. By Lemma 3.7, X has P_n dense for all n but is not transitive neither weakly blending. Since X is not transitive then it is not $l.e.o$, since transitivity is weaker than $l.e.o$. Therefore, on SFT, the property of P_n dense for all n does not imply $l.e.o$ neither weakly blending. ☐

Weakly blending is also weaker than those five chaos characterizations.

Theorem 3.9. *On SFT, $l.e.o$ implies weakly blending.*

Proof. By Theorem 2.2, $l.e.o$ implies weakly blending. ☐

The following counterexample shows that the converse Theorem 3.9 is not true.

Example 3.10. Let $X = X_{\mathcal{F}} = \{0\overline{111}, \overline{111}\}$ be a shift space with forbidden blocks $\mathcal{F} = \{00, 10\}$.

Lemma 3.11. *The shift space X as defined in Example 3.10 is weakly blending but neither transitive nor has dense periodic points.*

Proof. Since $X = \{0\overline{111}, \overline{111}\}$, then the only open sets are the $X, \phi, \overline{111}$ and $0\overline{111}$. Since $0\overline{111} \notin \sigma^n(\overline{111})$ for all integer n, then X is not transitive. However, since,

$$\sigma^n(\overline{111}) \cap \sigma^n(0\overline{111}) = \{\overline{111}\}$$

Then, X is strongly blending, and then weakly blending. The periodic points are not dense since $\{0\overline{111}\}$ does not have any periodic point. ☐

Theorem 3.12. *On SFT, weakly blending does not imply* $l.e.o$ *neither* P_n *dense for all* n.

Proof. Let us consider the SFT X as defined in Example 3.10. By Lemma 3.11, we showed that X is weakly blending, but neither transitive nor has dense periodic points. Therefore, X does not have the property of P_n dense for all n neither $l.e.o$, since transitivity is weaker than $l.e.o$. Hence, on shift of finite type, weakly blending does not imply $l.e.o$ neither P_n dense for all n. ☐

4. Implication Relation between Chaos Characterizations on Shift of Infinite Type

In this section, we present two examples of shift of infinite type (SIFT) to show that the equivalence of five chaos properties (i.e., totally transitive, topologically mixing, specification property, $l.e.o$, and Devaney chaos is not true for SIFT.

Example 4.1. The increasing gap shift $\Sigma(\mathrm{I})$ is an S −gap shift with

$$S = I = \{a_n : a_0 = 1, a_n = a_{n-1} + n, \forall n \in \mathbb{N}\} \subset \mathbb{N}$$

So, $\Sigma(I)$ is defined to be the set of all binary sequences for which 1's occurs infinitely often and the number of 0's between successive occurrences of 1 is an integer in I. Since $\Sigma(I)$ has infinitely many forbidden blocks, so it is a shift of infinite type.

Lemma 4.2. *The increasing gap shift,* $\Sigma(I)$ *is totally transitive, topologically mixing, Devaney chaotic but does not have the specification property nor* $l.e.o$ *(Kamarudin et al. 2019).*

Example 4.3. The shift space Σ_* is defined by;

$$\Sigma_* = \{(x_0, x_1, \ldots) \in \Sigma_* : x_i = 0, x_j = 2 \Rightarrow |i-j| \neq 2^p \ \forall p = 0,1,\ldots\}.$$

So Σ_* is the set of all infinite sequences of 0's, 1's and 2's for which no 0 and 2 are separated by a distance of 2^p for any number p. It is clear that Σ_* is shift of infinite type.

Lemma 4.4. *The* Σ_* *is totally transitive, but not topologically mixing,* $l.e.o$*, neither has the* Sp *(Crannell 1998).*

Theorem 4.5. *On SIFT, totally transitive is not equivalent to topologically mixing, specification property and* $l.e.o$.

Proof. Let us consider Σ_* as defined in Example 4.3. By Lemma 4.4, Σ_* is totally transitive but not topologically mixing, $l.e.o$ neither has the specification property. ☐

Theorem 4.6. *On SIFT, topologically mixing is not equivalence to totally transitive.*

Proof. Let us consider Σ_* as defined in Example 4.3. By Lemma 4.4, $\Sigma_*$is totally transitive but not topologically mixing. □

Theorem 4.7. *On SIFT, the specification property is not equivalence to totally transitive.*

Proof. Let us consider $\Sigma(I)$ as defined in Example 4.1. By Lemma 4.2, $\Sigma(I)$ is totally transitive but does not have the specification property. □

Theorem 4.8. *On SIFT, $l.e.o$ is not equivalence to totally transitive, topologically mixing, and Devaney chaos.*

Proof. Let us consider the increasing gap shift $\Sigma(I)$ as defined in Example 4.1. By Lemma 4.2, $\Sigma(I)$ is totally transitive, topologically mixing, and Devaney chaotic, but not $l.e.o$. □

Theorem 4.9. *On SIFT, Devaney chaotic is not equivalent to $l.e.o$.*

Proof. Let us consider the increasing gap shift $\Sigma(I)$ as defined in Example 4.1. By Lemma 4.2, $\Sigma(I)$ is Devaney chaotic but not $l.e.o$. □

5. Conclusion

Hierarchy diagrams in Figures 1 and 2 completely summarize all implication relations between seven chaos characterizations on general compact space and SFT, respectively. From other perspective, they give the order of the seven chaos characterizations according to the ability to implies other chaos characterizations. Existing results in various previous studies contributed to the process. To complete the process, we have proved some theorems and provided some counterexamples to verify other remaining implication relations. A surprising finding in this work is the differences of implication relations on SFT and SIFT.

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