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ON A SUBCLASS OF MEROMORPHIC FUNCTION DEFINED BY NEW DIFFERENTIAL OPERATOR INVOLVING THE MITTAG - LEFFLER FUNCTION

(Subkelas Fungsi Meromorfi Tertakrif oleh Pengoperasi Pembeza Baharu Melibatkan Fungsi Mittag-Leffler)

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ABSTRACT

In this work, we define a new differential operator with the multiplier transformation containing the Mittag - Leffler function. The multiplier transformation generates a new subclass of meromorphic function. Properties like coefficient estimates, growth and distortion, radii of starlikeness and convexity, and convex linear combination are given. Finally, the extreme points are also discussed.

Keywords: meromorphic function; differential operator; multiplier transformation; Hadamard product; Mittag - Leffler function

ABSTRAK

Dalam kajian ini, pengoperasi pembeza baharu tertakrif oleh jelmaan berganda mengandungi fungsi Mittag-Leffler. Jelmaan berganda ini membina subkelas fungsi meromorfi baharu. Sifat seperti pekali ketaksamaan, pertumbuhan dan erotan, jejari kebakbintangan dan kecembungan, dan kombinasi linear cembung diberi. Akhir sekali, titik ekstrim juga dibincangkan.

Kata kunci: fungsi meromorfi; pengoperasi pembeza; jelmaan berganda; hasil darab Hadamard; fungsi Mittag-Leffler

1. Introduction

Let $\sum$ be the class of functions written as

$$f(\zeta) = \frac{1}{\zeta} + \sum_{j=1}^{\infty} a_j \zeta^j, \quad a_j \in \mathcal{C}, \tag{1}$$

analytic in the punctured open unit disc $\Omega^* = \{\zeta : \zeta \in \mathcal{C}, : 0 < |\zeta| < 1\} = \Omega/\{0\}$. Let us denote $\sum^*(\varrho)$ and $\sum_k(\varrho)$, the subclasses of ϱ that are respectively, meromorphically starlike and convex functions of order ϱ. A function f of the form (1) is said to be in the class $\sum^*(\varrho)$ if it satisfies

$$\text{Re}\left(-\frac{\zeta f^{'}(\zeta)}{f(\zeta)}\right) > \varrho, \ (\zeta \in \Omega^*), \quad 0 \leq \varrho < 1$$

and $f \in \sum_k(\varrho)$ if it satisfies

$$\text{Re}\left(-\left(1 + \frac{\zeta f^{''}(\zeta)}{f^{'}(\zeta)}\right)\right) > \varrho, \ (\zeta \in \Omega^*), \quad 0 \leq \varrho < 1. \tag{2}$$

Let $g \in \Sigma$ be given by

$$g(\zeta) = \frac{1}{\zeta} + \sum_{j=1}^{\infty} b_j \zeta^j, \quad b_j \in \mathcal{C}. \tag{3}$$

The Hadamard product of the functions f and g is defined by $(f * g)$, as given below (Attiya 2016) :

$$(f * g)(\zeta) = \frac{1}{\zeta} + \sum_{j=1}^{\infty} a_j b_j \zeta^j. \tag{4}$$

A number of subclasses of meromorphic functions have been defined and discussed in the past. In particular, the subclasses $\mathrm{MK}_{1,m}^{\alpha}(\lambda, \alpha)$, $0 \leq \alpha < 1$ and $0 \leq \lambda < 1$ by (Cho *et al.* 2004), and $W(\alpha, \varphi)$, $0 \leq \alpha < 1$ and $0 < \varphi \leq 1$ by (Cho & Kim 2003). Given two functions $f(\zeta)$ and $g(\zeta)$ analytic in Ω, the function $f(\zeta)$ is subordinate to $g(\zeta)$, written $f \prec g$ or $f(\zeta) \prec g(\zeta)$ $(\zeta \in \Omega)$ if there exists a Schwarz function $W(\zeta)$ in Ω, such that $f(\zeta) = g(W(\zeta)),\ (\zeta \in \Omega)$ with $W(0) = 0$ and $|W(\zeta)| < 1, (\zeta \in \Omega)$.

For $r \in \mathbb{N}$ and $f \in \sum$ of the form Eq. (1), Cho and Srivastava (2004) and Cho and Kim (2003) studied the well-known multiplier transformation operator $I_1(r, \sigma)$, by the following series:

$$I_1(r, \sigma) f(\zeta) = \frac{1}{\zeta} + \sum_{j=2}^{\infty} \left(\frac{j + \sigma}{1 + \sigma} \right)^r a_j \zeta^j, \quad (\sigma \geq 0), \tag{5}$$

so that, obviously, $I_1(r_2, \sigma)(I_1(r_1, \sigma) f(\zeta)) = I_1(r_1 + r_2, \sigma) f(\zeta),\ r_1, r_2 \in \mathbb{N}$.

To begin with, we first state the well-known Mittag–Leffler function $E_\alpha(\zeta)$ introduced by Mittag-Leffler himself (Mittag-Leffler 1902, 1903), and Wiman's generalisation $E_{\alpha,\varphi}(\zeta)$ (Wiman 1905) given respectively as follows:

$$E_\alpha(\zeta) = \sum_{j=0}^{\infty} \frac{\zeta^j}{\Gamma(\alpha j + 1)} \tag{6}$$

and

$$E_{\alpha,\varphi}(\zeta) = \sum_{j=0}^{\infty} \frac{\zeta^j}{\Gamma(\alpha j + \varphi)}, \tag{7}$$

where $\alpha, \varphi \in \mathcal{C}$, $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\varphi) > 0$.

Further details regarding the Mittag–Leffler functions can be found in (Attiya 2016; Srivastava & Tomovski 2009; Gupta & Debnath 2007). Most of the work related to Mittag–Leffler functions are concentrating on the convexity, close-to-convexity and starlikeness. Bansal and Prajapat (Bansal & Prajapat 2016) studied on the $E_{\alpha,\varphi}(\zeta)$ of Mittag–Leffler function, while (Răducanu 2017) studied the partial sums for $E_{\alpha,\varphi}(\zeta)$. We note that the function given by (7) is not within the class Σ. Thus, the function is normalised in the following manner:

$$\Omega_{\alpha,\varphi}(\zeta) = \zeta^{-1}\Gamma(\varphi) E_{\alpha,\varphi}(\zeta) = \zeta^{-1} + \sum_{j=0}^{\infty} \frac{\Gamma(\varphi)}{\Gamma(\alpha(j+1) + \varphi)} \zeta^j, (\zeta \in \Omega^*). \tag{8}$$

Having the function $\Omega_{\alpha,\varphi}(\zeta)$ from Eq. (8), $I_1(r,\sigma)$ from Eq. (5) and making use of the Hadamard product for $f \in \Sigma$, we define a new linear differential operator $S^{\alpha}_{\varphi}[r,\sigma,\lambda]$, $0 \le \lambda \le 1$ on Σ as follows:

$$S^{\alpha}_{\varphi}[r,\sigma,\lambda]f(\zeta) = (1-\sigma)(I_1(r,\sigma)f(\zeta) * \Omega_{\alpha,\varphi}(\zeta) + \lambda\zeta((I_1(r,\sigma)f(\zeta) * \Omega_{\alpha,\varphi}(\zeta))'. \quad (9)$$

If $f \in \Sigma$, then from (Răducanu 2017) we deduce that

$$S^{\alpha}_{\varphi}[r,\sigma,\lambda]f(\zeta) = \frac{1}{\zeta} + \sum_{j=1}^{\infty}[1+(j-1)\lambda]^k(\frac{j+\sigma}{1+\sigma})^r \Omega_{j+1}(\alpha,\varphi)a_j\zeta^j, \quad (10)$$

where

$$\Omega_{j+1}(\alpha,\varphi) = \frac{\Gamma(\varphi)}{\Gamma(\alpha(j+1)+\varphi)}. \quad (11)$$

For a function $f \in \Sigma$, we define the following operator:

$$I^0(S^{\alpha}_{\varphi}[r,\sigma,\lambda]f(\zeta)) = f(\zeta),$$
$$I^1(S^{\alpha}_{\varphi}[r,\sigma,\lambda]f(\zeta)) = (1-\lambda)(S^{\alpha}_{\varphi}[r,\sigma,\lambda]f(\zeta)) + \lambda\zeta(S^{\alpha}_{\varphi}[r,\sigma,\lambda]f(\zeta))' + \frac{2}{\zeta}$$
$$I^2(S^{\alpha}_{\varphi}[r,\sigma,\lambda]f(\zeta)) = (1-\lambda)I^1(S^{\alpha}_{\varphi}[r,\sigma,\lambda]f(\zeta)) + \lambda\zeta(I^1(S^{\alpha}_{\varphi}[r,\sigma,\lambda]f(\zeta)))' + \frac{2}{\zeta},$$

and in general:

$$I^k(S^{\alpha}_{\varphi}[r,\sigma,\lambda]f(\zeta)) = (1-\lambda)I^{k-1}(S^{\alpha}_{\varphi}[r,\sigma,\lambda]f(\zeta)) + \lambda\zeta(I^{k-1}(S^{\alpha}_{\varphi}[r,\sigma,\lambda]f(\zeta)))' + \frac{2}{\zeta},$$
$$I^k((S^{\alpha}_{\varphi}[r,\sigma,\lambda]f(\zeta)) = \frac{1}{\zeta} + \sum_{j=1}^{\infty}[1+(j-1)\lambda]^k(\frac{j+\sigma}{1+\sigma})^r \Omega_{j+1}(\alpha,\varphi)a_j\zeta^j, \quad k = 1,2,\ldots .$$

By specialising the parameters of $I^k((S^{\alpha}_{\varphi}[r,\sigma,\lambda]f(\zeta))$, we get various operators as follows.

i) For $\lambda = 1, \varphi = 1, \alpha = 0, r = 0$ we get the operator studied by El-Ashwah and Aouf (El-Ashwah & Aouf 2009).

ii) For $\varphi = 1, \alpha = 0, r = 0, k = 0$, we get the operator studied by (Cho & Srivastava 2003) and (Cho & Kim 2003).

iii) For $\varphi = 1, \alpha = 0, r = 0$, we get the operator studied by (Challab & Darus 2016).

iv) For $k = 0, r = 0$, we obtain the operator presented by (Ameer *et al.* 2022).

Many authors have concentrated on subclasses of meromorphic function, for example, (Challab *et al.* 2017; Elrifai *et al.* 2012; Lashin 2010; Liu & Srivastava 2004; Elhaddad & Darus 2019) and others.

Using the operator $I^k((S^{\alpha}_{\varphi}[r,\sigma,\lambda]f(\zeta))$, we say that a function $f \in \Sigma$ belongs to $\mathrm{MK}^{\alpha,k}_{\varphi}(r,\sigma,\lambda,B,E,q)$. The class $\mathrm{MK}^{\alpha,k}_{\varphi}(r,\sigma,\lambda,B,E,q)$ is defined as follows:

Definition 1.1. For $-1 \le E < B \le 1$, $r \in \mathbb{N}$, $0 \le \lambda \le 1$, $\sigma \ge 0$, $\alpha, \varphi, q \in \mathcal{C}$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\varphi) > 0$ and $\mathrm{Re}(q) > 0$, the function $f \in \Sigma$ is in the class $\mathrm{MK}^{\alpha,k}_{\varphi}(r,\sigma,\lambda,B,E,q)$ if it satisfies

$$1-\frac{1}{q}\left(\frac{\zeta(I^k((S_\varphi^\alpha[r,\sigma,\lambda]f(\zeta)))'}{I^k((S_\varphi^\alpha[r,\sigma,\lambda]f(\zeta))}+1\right)\prec\frac{1+B\zeta}{1+E\zeta} \tag{12}$$

or, equivalently, to:

$$\left|\frac{\frac{\zeta(I^k((S_\varphi^\alpha[r,\sigma,\lambda]f(\zeta)))'}{I^k((S_\varphi^\alpha[r,\sigma,\lambda]f(\zeta))}+1}{E\frac{\zeta(I^k((S_\varphi^\alpha[r,\sigma,\lambda]f(\zeta)))'}{I^k((S_\varphi^\alpha[r,\sigma,\lambda]f(\zeta))}+(B-E)q+E}\right|<1. \tag{13}$$

Let Σ^* denote the subclass of Σ consisting of function of the form:

$$f(\zeta)=\frac{1}{\zeta}+\sum_{j=1}^{\infty}|a_j|\zeta^j. \tag{14}$$

Now, we define the class $\mathrm{MK}_\varphi^{\alpha,k,*}(r,\sigma,\lambda,B,E,q)$ by

$$\mathrm{MK}_\varphi^{\alpha,k,*}(r,\sigma,\lambda,B,E,q)\quad=\quad\mathrm{MK}_\varphi^{\alpha,k}(r,\sigma,\lambda,B,E,q)\cap\Sigma^*.$$

2. Main Result

This first part provides sufficient conditions under which Eq. (14) is indeed in the class $\mathrm{MK}_\varphi^{\alpha,k,*}(r,\sigma,\lambda,B,E,q)$. We also study the linear combinations, bounds for growth and distortion, closure theorems and extreme points.

Following the same method used by many authors, including by Elhaddad and Darus (Elhaddad & Darus 2019), and Liu and Strivastava (Liu & Srivastava 2004), we obtain the following result.

Theorem 2.1. *Let the function $f(\zeta)$ be of the form Eq. (14). Then $f\in\mathrm{MK}_\varphi^{\alpha,k,*}(r,\sigma,\lambda,B,E,q)$ if and only if*

$$\sum_{j=1}^{\infty}[(j+1)(1-E)-|q|(B-E)][1+(j-1)\lambda]^k\left(\frac{j+\sigma}{1+\sigma}\right)^r\Omega_{j+1}(\alpha,\varphi)|a_j|\le|q|(B-E). \tag{15}$$

Proof. Suppose that $f(\zeta)\in\mathrm{MK}_\varphi^{\alpha,k,*}(r,\sigma,\lambda,B,E,q)$ with $f(\zeta)$ of the form Eq. (14), then from Eq. (13) we have

$$\left|\frac{\zeta(I^k((S_\varphi^\alpha[r,\sigma,\lambda]f(\zeta)))'+(I^k((S_\varphi^\alpha[r,\sigma,\lambda]f(\zeta)))}{E\zeta(I^k((S_\varphi^\alpha[r,\sigma,\lambda]f(\zeta)))'+[(B-E)q+E](I^k((S_\varphi^\alpha[r,\sigma,\lambda]f(\zeta)))}\right|$$

$$=\left|\frac{\sum_{j=1}^{\infty}(j+1)[1+(j-1)\lambda]^k(\frac{j+\sigma}{1+\sigma})^r\Omega_{j+1}(\alpha,\varphi)|a_j|\zeta^{j+1}}{q(B-E)+\sum_{j=1}^{\infty}[E(j+1)+q(B-E)][1+(j-1)\lambda]^k(\frac{j+\sigma}{1+\sigma})^r\Omega_{j+1}(\alpha,\varphi)|a_j|\zeta^{j+1}}\right|<1.$$

We know that $|\operatorname{Re}(\zeta)|\le|\zeta|$, thus

$$\operatorname{Re}\left(\frac{\sum_{j=1}^{\infty}(j+1)[1+(j-1)\lambda]^k(\frac{j+\sigma}{1+\sigma})^r\Omega_{j+1}(\alpha,\varphi)|a_j|\zeta^{j+1}}{q(B-E)+\sum_{j=1}^{\infty}[E(j+1)+q(B-E)][1+(j-1)\lambda]^k(\frac{j+\sigma}{1+\sigma})^r\Omega_{j+1}(\alpha,\varphi)|a_j|\zeta^{j+1}}\right)<1. \tag{16}$$

Consider real values of ζ, and take $\zeta = s(0 \leq s < 1)$. For $s = 0$, the denominator is positive and so is positive for all $s(0 < s < 1)$. Letting $\zeta \longrightarrow 1-$, Eq. (16) gives

$$\sum_{j=1}^{\infty}[(j+1)(1-E)-|q|(B-E)][1+(j-1)\lambda]^k(\frac{j+\sigma}{1+\sigma})^r\Omega_{j+1}(\alpha,\varphi)|a_j| \leq |q|(B-E).$$

Thus, we obtain Eq. (15) of Theorem 2.1.

Conversely, we observe that

$$\left|\frac{\frac{\zeta(I^k((S_{\varphi}^{\alpha}[r,\sigma,\lambda]f(\zeta)))'}{I^k((S_{\varphi}^{\alpha}[r,\sigma,\lambda]f(\zeta))}+1}{E\frac{\zeta(I^k((S_{\varphi}^{\alpha}[r,\sigma,\lambda]f(\zeta)))'}{I^k((S_{\varphi}^{\alpha}[r,\sigma,\lambda]f(\zeta))}+(B-E)q+E}\right|$$

$$= \left|\frac{\zeta(I^k((S_{\varphi}^{\alpha}[r,\sigma,\lambda]f(\zeta)))'+(I^k((S_{\varphi}^{\alpha}[r,\sigma,\lambda]f(\zeta)))}{E\zeta(I^k((S_{\varphi}^{\alpha}[r,\sigma,\lambda]f(\zeta)))'+[(B-E)q+E](I^k((S_{\varphi}^{\alpha}[r,\sigma,\lambda]f(\zeta)))}\right|$$
$$= \left|\frac{\sum_{j=1}^{\infty}(j+1)[1+(j-1)\lambda]^k(\frac{j+\sigma}{1+\sigma})^r\Omega_{j+1}(\alpha,\varphi)|a_j|}{q(B-E)+\sum_{j=1}^{\infty}[E(j+1)+q(B-E)][1+(j-1)\lambda]^k(\frac{j+\sigma}{1+\sigma})^r\Omega_{j+1}(\alpha,\varphi)|a_j|}\right| \leq 1$$

by making use of Eq. (15) and set $|\zeta| = 1$. Thus, we have $f \in \mathrm{MK}_{\varphi}^{\alpha,k,*}(r,\sigma,\lambda,B,E,q)$ by maximum modulus theorem. □

Corollary 2.2. *If the function f of the form Eq. (14) is in the class $\mathrm{MK}_{\varphi}^{\alpha,k,*}(r,\sigma,\lambda,B,E,q)$, then*

$$|a_j| \leq \frac{|q|(B-E)}{[(j+1)(1-E)-|q|(B-E)][1+(j-1)\lambda]^k(\frac{j+\sigma}{1+\sigma})^r\Omega_{j+1}(\alpha,\varphi)}, \qquad j \geq 1.$$

The result is sharp for the function

$$f(\zeta) = \frac{1}{\zeta} + \frac{|q|(B-E)}{[(j+1)(1-E)-|q|(B-E)][1+(j-1)\lambda]^k(\frac{j+\sigma}{1+\sigma})^r\Omega_{j+1}(\alpha,\varphi)}\zeta^j, \quad j \geq 1. \tag{17}$$

The bounds for growth and distortion are given respectively as follows:

Theorem 2.3. *If a function f given by (14) is in the class $\mathrm{MK}_{\varphi}^{\alpha,k,*}(r,\sigma,\lambda,B,E,q)$, then for $|\zeta| = p < 1$, we get:*

$$\frac{1}{p} - \frac{|q|(B-E)}{[2(1-E)-|q|(B-E)](\frac{2+\sigma}{1+\sigma})^r\Omega_2(\alpha,\varphi)}p \leq |f(\zeta)| \leq \frac{1}{p} + \frac{|q|(B-E)}{[2(1-E)-|q|(B-E)](\frac{2+\sigma}{1+\sigma})^r\Omega_2(\alpha,\varphi)}p,$$

and

$$\frac{1}{p^2} - \frac{|q|(B-E)}{[2(1-E)-|q|(B-E)](\frac{2+\sigma}{1+\sigma})^r\Omega_2(\alpha,\varphi)} \leq |f'(\zeta)| \leq \frac{1}{p^2} + \frac{|q|(B-E)}{[2(1-E)-|q|(B-E)](\frac{2+\sigma}{1+\sigma})^r\Omega_2(\alpha,\varphi)},$$

for

$$2(1-E)-|q|(B-E)>0.$$

Proof. Let $f(\zeta)\in \mathrm{MK}_{\varphi}^{\alpha,k,*}(r,\sigma,\lambda,B,E,q)$. Then from the Theorem 2.1, we can get:

$$[2(1-E)-|q|(B-E)](\tfrac{2+\sigma}{1+\sigma})^r\Omega_2(\alpha,\varphi)\sum_{j=1}^{\infty}\tfrac{j!}{(j-1)!}|a_j|$$
$$\leq [(j+1)(1-E)-|q|(B-E)][1+(j-1)\lambda]^k(\tfrac{j+\sigma}{1+\sigma})^r\Omega_{j+1}(\alpha,\varphi)\leq |q|(B-E).$$

Thus

$$\sum_{j=1}^{\infty}|a_j|\leq \frac{|q|(B-E)}{[2(1-E)-|q|(B-E)](\frac{2+\sigma}{1+\sigma})^r\Omega_2(\alpha,\varphi)}. \tag{18}$$

Therefore,

$$|f(\zeta)|\leq \frac{1}{|\zeta|}+|\zeta|\sum_{j=1}^{\infty}|a_j|\leq \frac{1}{|\zeta|}+\frac{|q|(B-E)}{[2(1-E)-|q|(B-E)](\frac{2+\sigma}{1+\sigma})^r\Omega_2(\alpha,\varphi)}|\zeta|,$$

and

$$|f(\zeta)|\geq \frac{1}{|\zeta|}-|\zeta|\sum_{j=1}^{\infty}|a_j|\geq \frac{1}{|\zeta|}-\frac{|q|(B-E)}{[2(1-E)-|q|(B-E)](\frac{2+\sigma}{1+\sigma})^r\Omega_2(\alpha,\varphi)}|\zeta|.$$

Now, by differentiating both sides of Eq. (14) with respect to ζ, we have:

$$|f^{'}(\zeta)|\leq \frac{1}{|\zeta|^2}+\sum_{j=1}^{\infty}|a_j|\leq \frac{1}{|\zeta|^2}+\frac{|q|(B-E)}{[2(1-E)-|q|(B-E)](\frac{2+\sigma}{1+\sigma})^r\Omega_2(\alpha,\varphi)},$$

and

$$|f^{'}(\zeta)|\geq \frac{1}{|\zeta|^2}-\sum_{j=1}^{\infty}|a_j|\geq \frac{1}{|\zeta|^2}-\frac{|q|(B-E)}{[2(1-E)-|q|(B-E)](\frac{2+\sigma}{1+\sigma})^r\Omega_2(\alpha,\varphi)}.$$

Next, we obtain the radii of meromorphic starlikeness and convexity of order σ for functions in the class $\mathrm{MK}_{\varphi}^{\alpha,k,*}(r,\sigma,\lambda,B,E,q)$. □

Theorem 2.4. *Let the function f given by Eq. (14) be in the class $\mathrm{MK}_{\varphi}^{\alpha,k,*}(r,\sigma,\lambda,B,E,q)$. Thus, we have:*
(i) A function f is meromorphically starlike of order ϱ in the disc $|\zeta|<p_1$, that is

$$\mathrm{Re}\left(-\frac{\zeta f^{'}(\zeta)}{f(\zeta)}\right)>\varrho,\ (|\zeta|<p_1, 0\leq\varrho<1),$$

where

$$p_1=inf_{j\geq 1}\left\{\frac{(1-\varrho)[(j+1)(1-E)-|q|(B-E)][1+(j-1)\lambda]^k(\frac{j+\sigma}{1+\sigma})^r\Omega_{j+1}(\alpha,\varphi)}{|q|(B-E)(j+\varrho)}\right\}. \tag{19}$$

(ii) A function f is meromorphically convex of order ϱ in the disc $|\zeta|<p_2$, that is

$$\mathrm{Re}\left(-\left(1+\frac{\zeta f''(\zeta)}{f'(\zeta)}\right)\right) > \varrho,\ \ (|\zeta| < p_2, 0 \le \varrho < 1),$$

and

$$p_2 = inf_{j\ge 1}\left\{\frac{(1-\varrho)[(j+1)(1-E)-|q|(B-E)][1+(j-1)\lambda]^k(\frac{j+\sigma}{1+\sigma})^r\Omega_{j+1}(\alpha,\varphi)}{j|q|(B-E)(j+\varrho)}\right\}^{\frac{1}{n+1}} \tag{20}$$

Proof. (i) From Eq. (14), we can get:

$$\left|\frac{\frac{\zeta f'(\zeta)}{f(\zeta)}+1}{\frac{\zeta f'(\zeta)}{f(\zeta)}-1+2\varrho}\right| \le \frac{\sum_{j=1}^{\infty}(j+1)|a_j||\zeta|^{j+1}}{2(1-\varrho)-\sum_{j=1}^{\infty}(j-1+2\varrho)|a_j||\zeta|^{j+1}}.$$

Then, we have

$$\left|\frac{\frac{\zeta f'(\zeta)}{f(\zeta)}+1}{\frac{\zeta f'(\zeta)}{f(\zeta)}-1+2\varrho}\right| \le 1 \qquad (0\le\varrho<1),$$

if

$$\sum_{j=1}^{\infty}(\frac{j+\sigma}{1-\sigma})|a_j||\zeta|^{j+1} \le 1. \tag{21}$$

By Theorem 2.1, the inequality Eq. (21) holds if

$$(\frac{j+\varrho}{1-\varrho})|\zeta|^{j+1} \le \frac{[(j+1)(1-E)-|q|(B-E)][1+(j-1)\lambda]^k(\frac{j+\sigma}{1+\sigma})^r\Omega_{j+1}(\alpha,\varphi)}{|q|(B-E)},$$

then

$$|\zeta| \le \{\frac{(1-\varrho)[(j+1)(1-E)-|q|(B-E)][1+(j-1)\lambda]^k(\frac{j+\sigma}{1+\sigma})^r\Omega_{j+1}(\alpha,\varphi)}{j|q|(B-E)(j+\varrho)}\}^{\frac{1}{j+1}}.$$

Thus from Eq. (19), we have $|\zeta| < p_1$.

(ii) To prove the second assertion of Theorem 2.3, from Eq. (14) we find that:

$$\left|\frac{\frac{\zeta f''(\zeta)}{f'(\zeta)}+2}{\frac{\zeta f''(\zeta)}{f'(\zeta)}+2\varrho}\right| \le \frac{\sum_{j=1}^{\infty}j(j+1)|a_j||\zeta|^{j+1}}{2(1-\sigma)-\sum_{j=1}^{\infty}(j-1+2\varrho)|a_j||\zeta|^{j+1}}. \tag{22}$$

Thus, the result:

$$\left|\frac{\frac{\zeta f''(\zeta)}{f'(\zeta)}+2}{\frac{\zeta f''(\zeta)}{f'(\zeta)}+2\varrho}\right| \le 1 \qquad (0\le\varrho<1),$$

if

$$\sum_{j=1}^{\infty}(j(\frac{j+\varrho}{1-\varrho}))|a_j||\zeta|^{j+1} \leq 1. \tag{23}$$

Hence, by Theorem 2.1, inequality Eq. (23) holds if:

$$j(\frac{j+\varrho}{1-\varrho})|\zeta|^{j+1} \leq \frac{[(j+1)(1-E)-|q|(B-E)][1+(j-1)\lambda]^k(\frac{j+\varrho}{1+\sigma})^r\Omega_{j+1}(\alpha,\varphi)}{|q|(B-E)},$$

then

$$|\zeta| \leq \left\{\frac{(1-\varrho)[(j+1)(1-E)-|q|(B-E)][1+(j-1)\lambda]^k(\frac{j+\varrho}{1+\varrho})^r\Omega_{j+1}(\alpha,\varphi)}{j|q|(B-E)(j+\varrho)}\right\}^{\frac{1}{j+1}}.$$

Thus we obtain $|\zeta| < p_2$, where p_2 is noted by Eq. (20).
The closure theorems and extreme points of the class $\mathrm{MK}_{\varphi}^{\alpha,k,*}(r,\sigma,\lambda,B,E,q)$ are determined next. □

Theorem 2.5. *The class* $\mathrm{MK}_{\varphi}^{\alpha,k,*}(r,\sigma,\lambda,B,E,q)$ *is closed under convex linear combinations.*

Proof. Assume that the functions

$$f(\zeta) = \frac{1}{\zeta} + \sum_{j=1}^{\infty}|a_j|\zeta^j \qquad (j=1,2)$$

are in $\mathrm{MK}_{\varphi}^{\alpha,k,*}(r,\sigma,\lambda,B,E,q)$. It is enough to show that the function h defined by
$h(\zeta) = (1-a)f_1(\zeta) + af_2(\zeta)$ $(0 \leq a \leq 1)$, is in the class $\mathrm{MK}_{\varphi}^{\alpha,k,*}(r,\sigma,\lambda,B,E,q)$ since

$$h(\zeta) = \frac{1}{\zeta} + \sum_{j=1}^{\infty}[(1-a)|a_{j,1}| + a|a_{j,2}|]\zeta^j \qquad (0 \leq a \leq 1).$$

In view of Theorem 2.1, we have:

$$[(j+1)(1-E)-|q|(B-E)][1+(j-1)\lambda]^k(\frac{j+\sigma}{1+\sigma})^r\Omega_{j+1}(\alpha,\varphi)((1-a)|a_{j,1}|+a|a_{j,2}|)$$

$$= (1-a)[(j+1)(1-E)-|q|(B-E)][1+(j-1)\lambda]^k(\frac{j+\sigma}{1+\sigma})^r\Omega_{j+1}(\alpha,\varphi)|a_{j,1}|+$$

$$a[(j+1)(1-E)-|q|(B-E)][1+(j-1)\lambda]^k(\frac{j+\sigma}{1+\sigma})^r\Omega_{j+1}(\alpha,\varphi)|a_{j,2}|$$

$$\leq (1-a)|q|(B-E) + a|q|(B-E) = |q|(B-E).$$

This shows that $h(\zeta) \in \mathrm{MK}_{\varphi}^{\alpha,k,*}(r,\sigma,\lambda,B,E,q)$. □

Theorem 2.6. *Let* $f_0(\zeta) = \frac{1}{\zeta}$ *and*

$$f_j(\zeta) = \frac{1}{\zeta} + \frac{|q|(B-E)}{[2(1-E) - |q|(B-E)](\frac{j+\sigma}{1+\sigma})^r \Omega_{j+1}(\alpha, \varphi)} \zeta^j, \qquad (j \geq 1).$$

Then $f \in \mathrm{MK}_{\varphi}^{\alpha,k,*}(r, \sigma, \lambda, B, E, q)$ *if and only if it can be written in the form*

$$f(\zeta) = \sum_{j=1}^{\infty} v_j f_j(\zeta) \tag{24}$$

where $v_j \geq 0$ *and* $\sum_{j=0}^{\infty} v_j = 1$.

Proof. Let the function $f(\zeta)$ be written in the form given by Eq. (24), then

$$f(\zeta) = \frac{1}{\zeta} + \sum_{j=1}^{\infty} v_j \frac{|q|(B-E)}{[2(1-E) - |q|(B-E)](\frac{j+\sigma}{1+\sigma})^r \Omega_{j+1}(\alpha, \varphi)} \zeta^j.$$

For this function, we have

$$[2(1-E) - |q|(B-E)](\frac{j+\sigma}{1+\sigma})^r \Omega_{j+1}(\alpha, \varphi) \times v_j \frac{|q|(B-E)}{[2(1-E) - |q|(B-E)](\frac{j+\sigma}{1+\sigma})^r \Omega_{j+1}(\alpha, \varphi)}$$

$$= \sum_{j=1}^{\infty} v_j |q|(B-E) = |q|(B-E)(1-v_0) = |q|(B-E) \leq |q|(B-E).$$

The condition Eq. (15) is satisfied. Thus ,$f \in \mathrm{MK}_{\varphi}^{\alpha,k,*}(r, \sigma, \lambda, B, E, q)$ since

$$|a_j| \leq \frac{|q|(B-E)}{[2(1-E) - |q|(B-E)](\frac{j+\sigma}{1+\sigma})^r \Omega_{j+1}(\alpha, \varphi)}, \qquad (j \geq 1).$$

We set

$$v_j = \frac{[2(1-E) - |q|(B-E)](\frac{j+\sigma}{1+\sigma})^r \Omega_{j+1}(\alpha, \varphi)}{|q|(B-E)} |a_j|, \qquad (j \geq 1),$$

and

$$v_0 = 1 - \sum_{j=1}^{\infty} v_j.$$

So it follows that

$$f(\zeta) = 1 - \sum_{j=0}^{\infty} v_j f_j(\zeta).$$

Theorem 2.6 is complete. □

3. Conclusions

In this particular work, we introduced a new meromorphic function subclass which is related to both the Mittag-Leffler function and the multiplier transformation function. The sufficient and necessary conditions concerning this subclass are given. Some properties like linear combinations, radii, and others were explored. For future research, we may look into different classes concerning symmetric points defined by the multiplier transformation and Mittag-Leffler functions.

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