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A Single-Step Approach with Seven Hybrid Points for the Solution of Stiff Differential Equations

(Pendekatan Langkah Tunggal dengan Tujuh Titik Hibrid untuk Penyelesaian Persamaan Pembezaan Kaku)

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ABSTRACT

This research presents a novel single-step hybrid block method with seven intra-step points that achieves ninth-order accuracy, providing an accurate and computationally efficient approach for solving first-order stiff differential equations. The method is designed to solve first-order stiff differential equations with high efficiency and precision while maintaining a constant step size throughout the computation. To further improve accuracy, Lagrange polynomial interpolation techniques are employed to approximate function values at selected points within each step. The fundamental properties of the proposed scheme are rigorously analysed to establish its mathematical validity. These analyses confirm that the method satisfies the essential conditions of consistency, stability, and convergence, thereby ensuring its reliability for applications. The proposed method performs effectively when applied to stiff and oscillatory differential equations. Comprehensive numerical experiments are conducted, and the results consistently demonstrate the robustness and effectiveness of the proposed method across various test problems. Furthermore, the findings indicate that the method often outperforms several existing numerical techniques in terms of both accuracy and computational efficiency.

Keywords: Block hybrid method; intra-step points; stiff equations; zero stability

ABSTRAK

Penyelidikan ini memberikan satu kaedah blok hibrid satu langkah yang baharu dengan tujuh titik intra-langkah yang mencapai ketepatan tertib kesembilan, sekali gus menyediakan pendekatan yang tepat dan cekap dari segi pengiraan untuk menyelesaikan persamaan pembezaan biasa tegar tertib pertama. Kaedah ini direka dengan teliti untuk menyelesaikan persamaan pembezaan tertib pertama dengan kecekapan dan ketepatan yang tinggi sambil mengekalkan saiz langkah yang malar sepanjang pengiraan. Bagi meningkatkan lagi ketepatan, teknik interpolasi digunakan untuk menganggar nilai fungsi pada titik terpilih dalam setiap langkah. Sifat asas skim yang dicadangkan dianalisis secara teliti bagi membuktikan kesahihan matematiknya. Analisis ini mengesahkan bahawa kaedah tersebut memenuhi syarat penting iaitu kekonsistenan, kestabilan dan penumpuan, sekali gus menjamin kebolehpercayaannya untuk aplikasi praktikal. Selain itu, kaedah ini menunjukkan kebolehsuaian yang tinggi, menjadikannya sesuai untuk pelbagai jenis permasalahan. Secara khusus, kaedah ini berprestasi dengan berkesan apabila digunakan pada persamaan pembezaan tegar dan berayun. Uji kaji berangka yang menyeluruh telah dijalankan dan keputusannya secara tekal menunjukkan keteguhan dan keberkesanan kaedah yang dicadangkan merentasi pelbagai masalah ujian. Tambahan pula, keputusan kajian menunjukkan bahawa kaedah ini sering mengatasi beberapa teknik berangka sedia ada dari segi ketepatan dan kecekapan pengiraan.

Kata kunci: Kaedah hibrid blok; kestabilan sifar; persamaan tegar; titik intra

INTRODUCTION

A significant advancement in the study of numerical methods was the advent of modern computers, which revolutionized the process of function evaluation and numerical computation. The early reliance on manual or mechanical calculation methods gave way to automated and highly precise computational techniques. Over time, the integration of computers and specialized software applications into numerical methods has become a fundamental component of modern numerical analysis. This transformation enabled researchers to handle complex mathematical problems that were previously intractable due to computational limitations. The development of powerful computing tools such as MATLAB, GNU Octave,

Python, and Mathematica has further accelerated this progress. These tools offer exceptional speed, reliability, and flexibility in performing numerical experiments and data analysis. They also allow for easy visualization and simulation, enhancing the interpretation of results and theoretical understanding. Overall, the combination of advanced computational resources and sophisticated numerical techniques has elevated numerical analysis into a central discipline in scientific and engineering research.

Studies have indicated that a wide range of numerical methods have been employed by researchers to solve stiff differential equations. Among these, the block hybrid method is widely regarded as one of the most effective approaches for achieving accurate numerical results. Over time, numerous techniques have been developed to approximate solutions to stiff systems, including the implicit hybrid block method, which integrates block functions with interpolation. The demand for more precise numerical approximations, particularly in cases where analytical solutions are unavailable, has motivated extensive research across disciplines such as science, engineering, and the social sciences, leading to the development of various innovative numerical methods. Therefore, this research will concentrate on the use of the hybrid block method in solving first-order ordinary differential equations of the form.

$$y' = \frac{d}{dt} y(t) = f(t,y), y(0) = y_0 \; t \in [0.1]. \qquad (1)$$

Note that $t_0 = 0 < t_1 < \cdots < t_{N-1} < 1$ be a set of equally spaced points on the given interval [0,1], where $t_j = a + jh, j = 0, \ldots N$ and $h = \frac{t_N - t_0}{N}$ is a constant step size. This study reviews key contributions from the literature, with a particular emphasis on numerical methods for solving equation (1), focusing on the special case of single-step methods. Block methods have received considerable attention in recent research as effective tools for solving ordinary differential equations, especially first-order initial value problems (IVPs), where their application is well-established. The emergence of hybrid block methods has further advanced this area, as the integration of single-step methods within the block framework significantly enhances the accuracy and efficiency of the numerical solutions.

Over the years, block methods have proven to be effective numerical techniques for integrating ordinary differential equations (ODEs) (Henrici 1962). To address this advancement, hybrid block methods were developed to evaluate solutions at both grid and off-grid points within each discretization interval (Shampine & Watts 1969). Hybrid block methods have the advantages of varying step sizes and utilizing data at off-grid points, most importantly the ability to overcome the Dahlquist barrier of zero-stability and to improve the accuracy of the block methods (Henrici 1962). Recently, a number of researchers, including AbdulAzeez (2024), Dhandapani et al. (2019), Duromola, Momoh and Akinmoladun (2022), Motsa (2022), Norhayati et al. (2019), Obarhua (2023), Olukunle and Okoro (2019) and Tassaddiq et al. (2022), have been growing emphasis on the development of single-step hybrid block methods. Moreover, Bothayna and Muhammed I. (2019) developed an optimized single-step hybrid block method for the numerical solution of first-order initial value problems. Meanwhile, Bisheh-Niasar and Ramos (2023) developed a five-point hybrid block method for solving ordinary differential equations, incorporating interpolation techniques to evaluate delayed terms that were undefined at the grid points.

The growing interest in single-step hybrid block methods stems from their straightforward derivation, ease of implementation in computer algorithms, and ability to reduce errors. Their popularity in solving stiff differential equations (SDEs) is further enhanced by strong stability properties arising from fixed-step discretization, as well as the capacity to employ small step sizes without significant error amplification from perturbations. In the present study, a novel single-step hybrid block method with seven intra-step points is introduced. The method achieves ninth-order accuracy, satisfies A-stability, consistency, and convergence, and provides an efficient, accurate, and computationally economical approach for solving stiff differential equations. Stiff differential equations were characterized by components that evolve at markedly different rates, some varying rapidly while others change slowly, making them challenging for conventional numerical methods. The implicit methods generally offer greater stability for stiff problems; recent research continues to develop new block methods that achieve superior accuracy compared to existing approaches.

STIFF DIFFERENTIAL EQUATIONS

Numerical solutions are available for differential equations, but certain types, particularly stiff equations, are inherently challenging to solve and demand specialized numerical methods. This difficulty, known as *stiffness,* has been extensively investigated in numerical analysis due to the significant challenges it presents in maintaining both stability and accuracy (Sekar & Nalini 2015). Stiffness occurs when some components of the solution decay much more rapidly than others. Such problems exhibit exact solutions that are inherently stable, yet their numerical approximations can be highly unstable. Stiffness may be characterized through several distinct criteria.

KEY CHARACTERISTICS OF STIFF DIFFERENTIAL EQUATIONS

(i) Multiple time scales: Some components of the solution decay or change much faster than others.
(ii) Rapid transients: The system may have fast-decaying terms that are numerically difficult to handle.

(iii)Explicit solvers fail: Explicit methods (like Runge-Kutta) become unstable unless the time step is extremely small.
(iv)Implicit methods preferred: These methods (like backward Euler) are stable even for large step sizes in stiff problems.

Definition 1 A stiff differential equation is one that requires specialized numerical methods for an efficient and stable solution due to the presence of widely varying time scales in the system.

MATERIALS AND METHODS

The adoption of single-step hybrid block methods has gained significant attention in numerical analysis due to their balance between simplicity, stability, and computational efficiency. These methods are particularly attractive because they require less derivation effort and produce algorithms that are straightforward to implement on digital computers, thereby minimising sources of numerical error. Moreover, their strong stability properties resulting from fixed-step discretization allow for the use of smaller step sizes without considerable error amplification from perturbations, making them reliable for solving a wide range of ordinary differential equations (ODEs) (Wang, Wang & Cao 2019). The figure herewith illustrates the specification of the proposed method, which is a single-step approach incorporating seven hybrid points. These points are strategically selected to ensure zero stability and minimize numerical errors. For this method, the hybrid points are shown in Figure 1.

DERIVATION OF THE HYBRID BLOCK METHOD

In this section, we develop an implicit hybrid block method to solve Equations (1). To derive the block method, we assume that the differential equation is in the following form $y' = f(t, y)$. Over a subinterval $[t_n, t_{n+1}]$, assume that the exact solution $p(t)$ is;

$$p(t) = \sum_{i=0}^{9} a_i t^i. \tag{2}$$

where $a_i \in \mathbb{R}$ are real unknown coefficients. Therefore

$$p'(t) = \sum_{i=1}^{9} i a_i t^{i-1}. \tag{3}$$

As the custom, y_i and $f_i = f(t_i, y_i)$ are approximations for the exact values $y(t_i)$ and $f(t_j, y(t_j))$, respectively. In order to determine the coefficients a_i, we impose the following conditions:

$$p(t_n) = y_n\,. \tag{4}$$

$$p'(t_{n+\sigma}) = f_{n+\sigma},\ \sigma = 0, r_1, r_2, r_3, r_4, r_5, r_6, r_7, 1. \tag{5}$$

$$\text{where } 0 < r_i < 1 \text{ for } i = 1,2, \ldots 7 \tag{6}$$

and $t_{n+r_i} = t_n + r_i h$, $i = 1,.., 7$ are internal step points. That is $h = \frac{1}{8} = 0.125$. To generate the hybrid collocation points, we consider arbitrary collocation points. Evaluating (2) at point t_n and (3) at points $[t_{n+r_i} = t_n + r_i h,\ i = 1, 2, \ldots 7]$. Thus, we get the following system herewith:

$$\begin{bmatrix}
1 & t_n & t_n^2 & t_n^3 & t_n^4 & t_n^5 & t_n^6 & t_n^7 & t_n^8 & t_n^9 \\
0 & 1 & 2t_n & 3t_n^2 & 4t_n^3 & 5t_n^4 & 6t_n^5 & 7t_n^6 & 8t_n^7 & 9t_n^8 \\
0 & 1 & 2t_{n+r_1} & 3t_{n+r_1}^2 & 4t_{n+r_1}^3 & 5t_{n+r_1}^4 & 6t_{n+r_1}^5 & 7t_{n+r_1}^6 & 8t_{n+r_1}^7 & 9t_{n+r_1}^8 \\
0 & 1 & 2t_{n+r_2} & 3t_{n+r_2}^2 & 4t_{n+r_2}^3 & 5t_{n+r_2}^4 & 6t_{n+r_2}^5 & 7t_{n+r_2}^6 & 8t_{n+r_2}^7 & 9t_{n+r_2}^8 \\
0 & 1 & 2t_{n+r_3} & 3t_{n+r_3}^2 & 4t_{n+r_3}^3 & 5t_{n+r_3}^4 & 6t_{n+r_3}^5 & 7t_{n+r_3}^6 & 8t_{n+r_3}^7 & 9t_{n+r_3}^8 \\
0 & 1 & 2t_{n+r_4} & 3t_{n+r_4}^2 & 4t_{n+r_4}^3 & 5t_{n+r_4}^4 & 6t_{n+r_4}^5 & 7t_{n+r_4}^6 & 8t_{n+r_4}^7 & 9t_{n+r_4}^8 \\
0 & 1 & 2t_{n+r_5} & 3t_{n+r_5}^2 & 4t_{n+r_5}^3 & 5t_{n+r_5}^4 & 6t_{n+r_5}^5 & 7t_{n+r_5}^6 & 8t_{n+r_5}^7 & 9t_{n+r_5}^8 \\
0 & 1 & 2t_{n+r_6} & 3t_{n+r_6}^2 & 4t_{n+r_6}^3 & 5t_{n+r_6}^4 & 6t_{n+r_6}^5 & 7t_{n+r_6}^6 & 8t_{n+r_6}^7 & 9t_{n+r_6}^8 \\
0 & 1 & 2t_{n+r_7} & 3t_{n+r_7}^2 & 4t_{n+r_7}^3 & 5t_{n+r_7}^4 & 6t_{n+r_7}^5 & 7t_{n+r_7}^6 & 8t_{n+r_7}^7 & 9t_{n+r_7}^8 \\
0 & 1 & 2t_{n+1} & 3t_{n+1}^2 & 4t_{n+1}^3 & 5t_{n+1}^4 & 6t_{n+1}^5 & 7t_{n+1}^6 & 8t_{n+1}^7 & 9t_{n+1}^8
\end{bmatrix}
\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \\ \alpha_9 \end{bmatrix}
=
\begin{bmatrix} y_n \\ f_n \\ f_{n+r_1} \\ f_{n+r_2} \\ f_{n+r_3} \\ f_{n+r_4} \\ f_{n+r_5} \\ f_{n+r_6} \\ f_{n+r_7} \\ f_{n+1} \end{bmatrix}$$

We get

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \\ \alpha_9 \end{bmatrix}
=
\begin{bmatrix} R_0 \\ R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \\ R_9 \end{bmatrix}
\begin{bmatrix} y_n \\ f_n \\ f_{n+r_1} \\ f_{n+r_2} \\ f_{n+r_3} \\ f_{n+r_4} \\ f_{n+r_5} \\ f_{n+r_6} \\ f_{n+r_7} \\ f_{n+1} \end{bmatrix}
=
\begin{bmatrix} R_0 \\ R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \\ R_9 \end{bmatrix}
B \rightarrow a_i = R_i(t_n, h)B$$

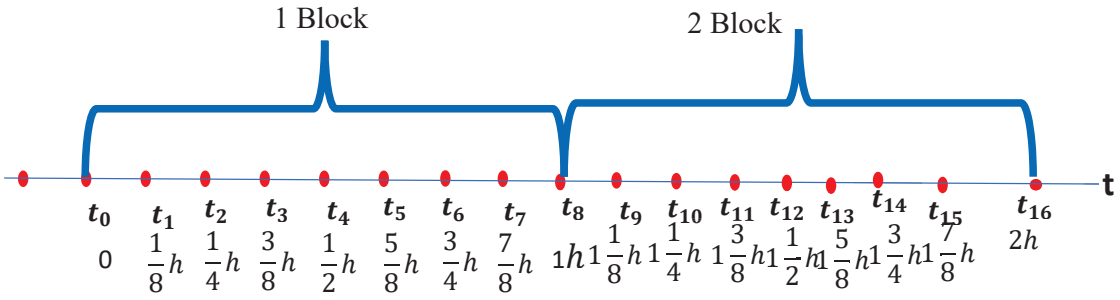


FIGURE 1. Single step block method with seven hybrid points

where $B = \begin{bmatrix} y_n \\ f_n \\ f_{n+r_1} \\ f_{n+r_2} \\ f_{n+r_3} \\ f_{n+r_4} \\ f_{n+r_5} \\ f_{n+r_6} \\ f_{n+r_7} \\ f_{n+1} \end{bmatrix}$ and $[\,R_i(t_n,h)]_{i=0\ldots9} = \begin{bmatrix} R_0 \\ R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \\ R_9 \end{bmatrix}$.

which must be solved to get the coefficients a_j, j = 0, 1, …, 9. We replace back into Equation (2), and after doing the change of variable $t = t_n + \lambda h$, the approximate solution $p(t_n + \lambda h)$ takes the form;

$$p = (t_n + \lambda h) = C_0 y_n + h(C_0(t)f_n + C_{r1}(t)f_{n+r1} + C_{r2}(t)f_{n+r2} + C_{r3}(t)f_{n+r3} + C_{r4}(t)f_{n+r4} + C_{r5}(t)f_{n+r5} + C_{r6}(t)f_{n+r6} + C_{r7}(t)f_{n+r7} + C_8(t)f_{n+1}.$$

$$p(t) = \sum_{i=0}^{9} a_i t^i = \sum_{i=0}^{9} R_i(t_n,h)Bt^i = \sum_{i=0}^{9} t^i R_i(t_n,h) \begin{bmatrix} y_n \\ f_n \\ f_{n+r_1} \\ f_{n+r_2} \\ f_{n+r_3} \\ f_{n+r_4} \\ f_{n+r_5} \\ f_{n+r_6} \\ f_{n+r_7} \\ f_{n+1} \end{bmatrix}.$$

Finally, we get the final form of Equation (7) as below:

$$p(t_n + \lambda h) = \sum_{i=0}^{9} \alpha_i (t_n + \lambda h)^i = \sum_{i=0}^{9} R_i(t_n,h)B(t_n + \lambda h)^i = \sum_{i=0}^{9} (t_n,\lambda h)^i R_i(t_n,h) \begin{bmatrix} y_n \\ f_n \\ f_{n+r_1} \\ f_{n+r_2} \\ f_{n+r_3} \\ f_{n+r_4} \\ f_{n+r_5} \\ f_{n+r_6} \\ f_{n+r_7} \\ f_{n+1} \end{bmatrix}.$$

$$p(t_n + \lambda h) = C_0 y_n + h(C_0 f_n + C_1 f_{n+r1} + C_2 f_{n+r2} + C_3 f_{n+r3} + C_4 f_{n+r4} + C_5 f_{n+r5} + C_6 f_{n+r6} + C_7 f_{n+r7} + C_8 f_{n+1}).$$

$$C_0 = \sum_{i=0}^{9} (t_n + \lambda h)^i R_{i,0}(t_n,h),$$

$hC_k = \sum_{i=0}^{9} (t_n + \lambda h)^i R_{i,k}(t_n,h)$, $k = 1,2, \ldots 9$.

where

$$\begin{cases} C_0 = 1 \\ C_n = \frac{1}{28350}h(1310720h^8 - 6635520h^7 + 14376960h^6 - 17418240h^5 + 12930624h^4 - 6055560h^3 + 1771860h^2 - 308205h + 28350), \\ C_{r_1} = \frac{32}{14175}h^2(163840h^7 + 806400h^6 - 1681920h^5 + 1932000h^4 - 1326528h^3 + 549675h^2 - 129870h + 14175), \\ C_{r_2} = -\frac{8}{14175}h^2(2293760h^7 - 10967040h^6 + 22026240h^5 - 24057600h^4 + 15411312h^3 - 5781195h^2 + 1173690h - 99225), \\ C_{r_3} = \frac{32}{14175}h^2(1146880h^7 + 5322240h^6 - 10298880h^5 + 10735200h^4 - 6483456h^3 + 2259495h^2 - 420630h + 33075), \\ C_{r_4} = -\frac{2}{2835}h^2(4587520h^7 - 20643840h^6 + 38522880h^5 - 38492160h^4 + 22161888h^3 - 7343280h^2 + 1305990h - 99225), \\ C_{r_5} = \frac{32}{14175}h^2(1146880h^7 + 4999680h^6 - 9008640h^5 + 8672160h^4 - 4810176h^3 + 1540665h^2 - 266490h + 19845), \\ C_{r_6} = -\frac{8}{14175}h^2(2293760h^7 - 9676800h^6 + 16865280h^5 - 15724800h^4 + 8476272h^3 - 2650725h^2 + 450030h - 33075), \\ C_{r_7} = \frac{32}{14175}(163840h^7 + 668160h^6 - 1128960h^5 + 1024800h^4 - 540288h^3 + 166005h^2 - 27810h + 2025), \\ C_{r_1} = \frac{h^2}{28350}(1310720h^7 - 5160960h^6 + 8478720h^5 - 7526400h^4 + 3898944h^3 - 1181880h^2 + 196020h - 14175). \end{cases}$$

Now, in order to obtain the main block method, we evaluate $p(t_n + \lambda h)$ at the values
$\lambda = r_i$, $i = 1,2, \ldots 7$, to get the following block hybrid method herewith:

$$\begin{cases} y_{n+r_1} = y_n + h(\frac{1070017}{29030400}f_n + \frac{2233547}{14515200}f_{n+r_1} - \frac{2302297}{14515200}f_{n+r_2} + + \frac{2797679}{14515200}f_{n+r_3} - \frac{31457}{181440}f_{n+r_4} + \frac{1573169}{14515200}f_{n+r_5} - \frac{645607}{14515200}f_{n+r_6} + \frac{156437}{14515200}f_{n+r_7} - \frac{33953}{29030400}f_{n+1}). \\ y_{n+r_2} = y_n + h(\frac{32377}{907200}f_n + \frac{22823}{113400}f_{r_1} - \frac{21247}{453600}f_{r_2} + \frac{15011}{113400}f_{r_3} - \frac{2903}{22680}f_{r_4} + \frac{9341}{113400}f_{r_5} - \frac{15577}{453600}f_{r_6} + \frac{953}{113400}f_{r_7} - \frac{119}{129600}f_{n+1}). \\ y_{n+r_3} = y_n + h(\frac{12881}{358400}f_n + \frac{35451}{179200}f_{r_1} + \frac{1719}{179200}f_{r_2} + \frac{39967}{179200}f_{r_3} - \frac{351}{2240}f_{r_4} + \frac{17217}{179200}f_{r_5} - \frac{7031}{179200}f_{r_6} + \frac{243}{25600}f_{r_7} - \frac{369}{358400}f_{n+1}). \\ y_{n+r_4} = y_n + h(\frac{4063}{113400}f_n + \frac{2822}{14175}f_{r_1} + \frac{61}{28350}f_{r_2} + \frac{4094}{14175}f_{r_3} - \frac{227}{2835}f_{r_4} + \frac{1154}{14175}f_{r_5} - \frac{989}{28350}f_{r_6} + \frac{122}{14175}f_{r_7} - \frac{107}{113400}f_{n+1}). \\ y_{n+r_5} = y_n + h(\frac{41705}{1161216}f_n + \frac{115075}{580608}f_{r_1} + \frac{3775}{580608}f_{r_2} + \frac{159175}{580608}f_{r_3} - \frac{125}{36288}f_{r_4} + \frac{85465}{580608}f_{r_5} + \frac{85465}{580608}f_{r_6} - \frac{24575}{580608}f_{r_7} + \frac{5725}{580608}f_{r_8} - \frac{175}{165888}f_{n+1}). \\ y_{n+r_6} = y_n + h(\frac{401}{11200}f_n + \frac{279}{1400}f_{r_1} + \frac{9}{5600}f_{r_2} + \frac{403}{1400}f_{r_3} - \frac{9}{280}f_{r_4} + \frac{333}{1400}f_{r_5} + \frac{79}{5600}f_{r_6} + \frac{9}{1400}f_{r_7} - \frac{9}{11200}f_{n+1}). \\ y_{n+r_7} = y_n + h(\frac{149527}{4147200}f_n + \frac{408317}{2073600}f_{r_1} + \frac{24353}{2073600}f_{r_2} + \frac{542969}{2073600}f_{r_3} + \frac{343}{25920}f_{r_4} + \frac{368039}{2073600}f_{r_5} + \frac{261023}{2073600}f_{r_6} + \frac{111587}{2073600}f_{r_7} - \frac{8183}{4147200}f_{n+1}). \\ y_{n+1} = y_n + h(\frac{989}{28350}f_n + \frac{2944}{14175}f_{r_1} - \frac{464}{14175}f_{r_2} + \frac{5248}{14175}f_{r_3} - \frac{454}{2835}f_{r_4} + \frac{5248}{14175}f_{r_5} - \frac{464}{14175}f_{r_6} + \frac{2944}{14175}f_{r_7} + \frac{989}{28350}f_{n+1}). \end{cases} \quad (8)$$

Now, let $r_i\, K = R_i$, we get the below transformation with $K = 8$:

$$\left[r_1 = \frac{1}{8}, r_2 = \frac{1}{4}, r_3 = \frac{3}{8}, r_4 = \frac{1}{2}, r_5 = \frac{5}{8}, r_6 = \frac{3}{4}, r_7 = \frac{7}{8}\right] \rightarrow K[r_1,r_2,r_3,r_4,r_5,r_6,r_7].$$
$$= 7[r_1,r_2,r_3,r_4,r_5,r_6,r_7] = [R_1,R_2,R_3,R_4,R_5,R_6,R_7] = [1,2,3,4,5,6,7].$$

Let $h = KH, y' = \frac{d}{dt}y = f(t,y)$, $y_n = y(t_n)$.

$$y(t_n + rh) = y(t_n + rKH) = y_{n+rK} = y_{n+R'} \quad rK = R \in z.$$

For $\frac{d}{dt}y = f(t,y)$, we get

$$y_{n+R_1} = y_n + KH(\frac{1070017}{29030400}f_n + \frac{2233547}{14515200}f_{n+R_1} - \frac{2302297}{14515200}f_{n+R_2} + + \frac{2797679}{14515200}f_{n+R_3} - \frac{31457}{181440}f_{n+R_4} + \frac{1573169}{14515200}f_{n+R_5} - \frac{645607}{14515200}f_{n+R_6} + \frac{156437}{14515200}f_{n+R_7} - \frac{33953}{29030400}f_{n+1}).$$

Or

$$a_{R1}y_{n+R1}+a_0y_n=KH(\beta_0f_n+\beta_{R_1}f_{n+R_1}+\beta_{R_2}f_{n+R_2}+\beta_{R_3}f_{n+R_3}+\beta_{R_4}f_{n+R_4}+\beta_{R_5}f_{n+R_5}+\beta_{R_6}f_{n+R_6}+\beta_{R_7}f_{n+R_7}+\beta_1f_{n+K}).$$

and we get

$$a_{R1}y_{n+R1}+a_0y_n=KH(\beta_0f_n+\beta_{r_1}f_{n+R_1}+\beta_{R_2}f_{n+R_2}+\beta_{R_3}f_{n+R_3}+\beta_{R_4}f_{n+R_4}+\beta_{R_5}f_{n+R_5}+\beta_{R_6}f_{n+R_6}+\beta_{R_7}f_{n+R_7}+\beta_1f_{n+K}).$$

Truncation error:

$$C_0=a_0+a_1+\cdots+a_j$$

$$C_1=a_1+2a_2+\cdots+ka_j-(\beta_0+\beta_1+\cdots+\beta_j)$$

$$\vdots \qquad\qquad \vdots$$

$$C_q=\frac{1}{q!}(\alpha_1+2^q\alpha_2+\cdots+k^q\alpha_j)-\frac{1}{(q-1)!}(\beta_1+2^{q-1}\beta_2+\cdots+k^{q-1}\beta_j),\qquad q\geq 2$$

Hence the truncation error is $C_qH^qy^{(q)}(t_n)$

$$H=\frac{1}{K}h\rightarrow C_qH^qy^{(q)}(t_n)=\frac{C_q}{K^q}h^qy^{(q)}(t_n).$$

Obviously, we find that $y_{n+Ri}=y_{n+i}, i=1, \ldots, 8$.
From (8) we get

$$y_{n+7}=y_n+KH[C_{R,1}f_n+C_{R,2}f_{n+1}+C_{R,3}f_{n+2}+C_{R,4}f_{n+3}+C_{R,5}f_{n+4}+C_{R,6}f_{n+5}+C_{R,7}f_{n+6}+C_{R,8}f_{n+7}]. \qquad (9)$$

where $a_0=-1, a_7=1, \beta_j=C_{k,j+1}$

Using the Lambert formulae (Lambert 1974), we get the linear multistep method scheme as

$$\sum_{j=0}^{k}\alpha_jy_{n+j}=KH\sum_{j=0}^{k}\beta_jf_{n+j}.$$

By using $\beta^*=K\beta=7\beta$ we get

$$\sum_{j=0}^{k}\alpha_jy_{n+j}=KH\sum_{j=0}^{k}\beta_jf_{n+j}=H\sum_{j=0}^{k}\beta_j^*f_{n+j},$$

$$a_{R1}y_{n+R1}+a_0y_n=H(\beta^*{}_0f_n+\beta^*{}_{r_1}f_{n+R_1}+\beta^*{}_{R_2}f_{n+R_2}+\beta^*{}_{R_3}f_{n+R_3}+\beta^*{}_{R_4}f_{n+R_4}+\beta^*{}_{R_5}f_{n+R_5}+\beta^*{}_{R_6}f_{n+R_6}+\beta^*{}_{R_7}f_{n+R_7}+\beta_1{}^*f_{n+K}),$$

ORDER AND ERROR CONSTANT OF THE METHOD

In this section, we present the major properties of Equation (8). To analyse the order of accuracy, we define the local truncation error and apply linear difference operators as follows:

$L_\sigma[y(t_n),h]=y(t_n+\sigma h)-h\Sigma_j\beta_{j,\sigma}y'(t_n+jh)$, $\sigma=R_1, R_2, R_3, R_4, R_5, R_6, R_7, 1$.

Where $\beta j,\sigma$ are the corresponding coefficients. We assume that $y(t)$ is sufficiently differentiable and expand $y(t_n+\sigma h)$ and $y'(t_n+jh)$, about t_n.

Given

$$\sum_{j=0}^{k}\alpha_jy_{n+j}=h\sum_{j=0}^{k}\beta_jf_{n+j}.$$

Let the linear difference operator

$$l[y(t);h]=\sum_{i=0}^{k}[\alpha_jy(t+jh)-h\beta_jy'(t+jh)].$$

Using Taylor's series, we get

$$l[y(t);h]=C_0y(t)+C_1hy^{(1)}(t)+C_2h^2y^{(2)}(t)+\cdots+C_ph^py^{(p)}(t)+\cdots$$

Definition 2. According to Lambert (1974), a linear multistep method has order p if all error constants up to Cp vanish and $C_{p+1}\neq 0$. The order and error constant of the proposed method (8) can be obtained from the following formula:

$$C_0=\sum_{j=0}^{k}\alpha_j=\begin{bmatrix}0\\0\end{bmatrix}$$

$$C_1=\sum_{j=0}^{k}(j\alpha_j-\beta_j)=\begin{bmatrix}0\\0\end{bmatrix}$$

$$C_2=\sum_{j=0}^{k}\left(\frac{j^2\alpha_j}{2!}-j\beta_j\right)=\begin{bmatrix}0\\0\end{bmatrix}$$

$$C_3=\sum_{j=0}^{k}\left(\frac{j^3\alpha_j}{3!}-\frac{j^2\beta_j}{2!}\right)=\begin{bmatrix}0\\0\end{bmatrix}$$

$$C_4=\sum_{j=0}^{k}\left(\frac{j^4\alpha_j}{4!}-\frac{j^3\beta_j}{3!}\right)=\begin{bmatrix}0\\0\end{bmatrix}$$

$$C_5=\sum_{j=0}^{k}\left(\frac{j^5\alpha_j}{5!}-\frac{j^4\beta_j}{4!}\right)=\begin{bmatrix}0\\0\end{bmatrix}$$

$$C_6=\sum_{j=0}^{k}\left(\frac{j^6\alpha_j}{6!}-\frac{j^5\beta_j}{5!}\right)=\begin{bmatrix}0\\0\end{bmatrix}$$

$$C_7=\sum_{j=0}^{k}\left(\frac{j^7\alpha_j}{7!}-\frac{j^6\beta_j}{6!}\right)=\begin{bmatrix}0\\0\end{bmatrix}$$

$$C_8=\sum_{j=0}^{k}\left(\frac{j^8\alpha_j}{8!}-\frac{j^7\beta_j}{7!}\right)=\begin{bmatrix}0\\0\end{bmatrix}$$

$$C_9=\sum_{j=0}^{k}\left(\frac{j^9\alpha_j}{9!}-\frac{j^8\beta_j}{8!}\right)=\begin{bmatrix}0\\0\end{bmatrix}$$

$$C_{10}=\left(\frac{8183}{1113255523123200},\frac{9}{1503238553600},\frac{25}{3848290697216},\frac{47}{7610145177600},\frac{25}{3848290697216},\frac{9}{1503238553600},\frac{8183}{1113255523123200}\right)^T.$$

This implies that the block method has order nine with error constant $C_{p+1}=C_{10}$. In the development of numerical

solutions for solving differential equations, it is well known that the order of a certain method plays an important role in determining its accuracy.

CONSISTENCY

Definition 3 A linear multistep method is regarded as consistent if its order is $p > 1$ (Lambert 1974). Consequently, the order of the proposed method is $9 > 1$ and by using the definition 3 above on the block method (8), this implies that the proposed method is consistent.

ZERO-STABILITY

Definition 4 A linear multistep method is said to be zero-stable if no root of the first characteristic polynomial ρ(Z) has modulus greater than one; it is simple if all roots have modulus one. A linear multistep technique must be zero-stable and consistent to converge (Lambert 1974). For the stability analysis, we can rearrange the hybrid block method (8) in the following matrix form:

$$A^1 Y_{n+1} = A^0 Y_n + h(B^0 F_n - B^1 F_{n+1}), \tag{9}$$

where

$Y_{n+1} = [y_{n+R_1}, y_{n+R_2}, y_{n+R_3}, y_{n+R_4}, y_{n+R_5}, y_{n+R_6}, y_{n+R_7}, y_{n+1}]^T$.

$Y_n = [y_{n+R_{1-1}}, y_{n+R_{2-1}}, y_{n+R_{3-1}}, y_{n+R_{4-1}}, y_{n+R_{5-1}}, y_{n+R_{6-1}}, y_{n+R_{7-1}}, y_n]^T$.

$F_{n+1} = [f_{n+R_1}, f_{n+R_2}, f_{n+R_3}, f_{n+R_4}, f_{n+R_5}, f_{n+R_6}, f_{n+R_7}, f_{n-1}]^T$.

$F_n = [f_{n+R_{1-1}}, f_{n+R_{2-1}}, f_{n+R_{3-1}}, f_{n+R_{4-1}}, f_{n+R_{5-1}}, f_{n+R_{6-1}}, f_{n+R_{7-1}}, f_n]^T$.

where

$$A^1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} A^0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

$B^1 =$

$$\begin{bmatrix} \frac{2233547}{14515200} & -\frac{2302297}{14515200} & \frac{2797679}{14515200} & -\frac{31457}{181440} & \frac{1573169}{14515200} & -\frac{645607}{14515200} & \frac{156437}{14515200} & -\frac{33953}{29030400} \\ \frac{22823}{113400} & -\frac{21247}{453600} & \frac{15011}{113400} & -\frac{2903}{22680} & \frac{9341}{113400} & -\frac{15577}{453600} & \frac{953}{113400} & -\frac{119}{129600} \\ \frac{35451}{179200} & \frac{1719}{179200} & \frac{39967}{179200} & -\frac{351}{2240} & \frac{17217}{179200} & -\frac{7031}{179200} & \frac{243}{25600} & -\frac{369}{358400} \\ \frac{2822}{14175} & \frac{61}{28350} & \frac{4094}{14175} & -\frac{227}{2835} & \frac{1154}{14175} & -\frac{989}{28350} & \frac{122}{14175} & -\frac{107}{113400} \\ \frac{115075}{580608} & \frac{3775}{580608} & \frac{159175}{580608} & -\frac{125}{36288} & \frac{85465}{580608} & -\frac{24575}{580608} & \frac{5725}{580608} & -\frac{175}{165888} \\ \frac{279}{1400} & \frac{9}{5600} & \frac{403}{1400} & -\frac{9}{280} & \frac{333}{1400} & -\frac{79}{5600} & \frac{9}{1400} & -\frac{9}{11200} \\ \frac{408317}{2073600} & \frac{24353}{2073600} & \frac{542969}{2073600} & -\frac{343}{25920} & \frac{368039}{2073600} & -\frac{261023}{2073600} & \frac{111587}{2073600} & -\frac{8183}{4147200} \\ \frac{2944}{14175} & -\frac{464}{14175} & \frac{5248}{14175} & -\frac{454}{2835} & \frac{5248}{14175} & -\frac{464}{14175} & \frac{2944}{14175} & \frac{989}{28350} \end{bmatrix}$$

$$B^0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1070017}{29030400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{32377}{907200} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{12881}{358400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4063}{113400} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{41705}{1161216} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{401}{11200} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{149527}{4147200} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{989}{28350} \end{bmatrix}$$

$f_{n+R_1} = f(x_n + Z\Delta t, y(t_n + Z\Delta t)) = f(t_{n+R_1}, y_{n+R_1}), y' = f(t, y).$

$$A^1 Y_{n+1} = A^0 Y_n + h(B^0 F_n + B^1 F_{n+1}).$$

Let $F_n = F_{n+1} = 0$,

$$A^1 Y_{n+1} = A^0 Y_n + h(B^0 F_n + B^1 F_{n+1}) \longrightarrow A^1 Y_{n+1} - A^0 Y_n = 0.$$

Let $y_i = Z^i$, we get

$$Y_{n+1} = [y_{n+R_1}, y_{n+R_2}, y_{n+R_3}, y_{n+R_4}, y_{n+R_5}, y_{n+R_6}, y_{n+R_7}, y_{n+1}]^T.$$

$$= [Z^{n+R_1}, Z^{n+R_2}, Z^{n+R_3}, Z^{n+R_4}, Z^{n+R_5}, Z^{n+R_6}, Z^{n+R_7}, Z^{n+1}]^T$$

$$A^1 Z Y_n - A^0 Y_n = 0 \longrightarrow (A^1 Z - A^0) Y_n = 0.$$

In order to get the nontrivial solution of Y_n, we need to confirm that $(A^1 Z - A^0)$ is nonsingular, hence, $det(A^1 Z - A^0) = 0$. We found that

$$p(Z) = det(A^1 Z - A^0) = Z \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= Z^7 (Z - 1) = 0 \longrightarrow Z_i = [0, 0, 0, 0, 0, 0, 0, 1,]. \tag{10}$$

Hence for the proposed method, we find $|P(Z)| \leq 1$ satisfying definition 4 above, hence the proposed method is zero-stable. The linear stability analysis of the method is performed by applying it to the test problem as follows:

$$y' = f(t, y) = \lambda y,\ Re(\lambda) < 0. \tag{11}$$

Thus

$F_{n+1} = [f_{n+R_1}, f_{n+R_2}, f_{n+R_3}, f_{n+R_4}, f_{n+R_5}, f_{n+R_6}, f_{n+R_7}, f_{n+1}]^T$.

$= [\lambda y_{n+R_1}, \lambda y_{n+R_2}, \lambda y_{n+R_3}, \lambda y_{n+R_4}, \lambda y_{n+R_5}, \lambda y_{n+R_6}, \lambda y_{n+R_7}, \lambda y_{n+1}]^T$.

$= \tau\ [y_{n+R_1}, y_{n+R_2}, y_{n+R_3}, y_{n+R_4}, y_{n+R_5}, y_{n+R_6}, y_{n+R_7}, y_{n+1}]^T = \lambda Y_{n+1}$.

Substituting

$$y' = \tau y \text{ in } A^0 Y_n + h(B^0 F_n + B^1 F_{n+1}),$$

we obtain

$$A^1 Y_{n+1} = A^0 Y_n + h(B^0 F_n + B^1 F_{n+1}) \rightarrow A^1 Y_{n+1} = A^0 Y_n + h(B^0 \lambda Y_n + B^1 \lambda Y_{n+1})$$

$$A^1 Y_{n+1} - hB^1 \lambda Y_{n+1} = A^0 Y_n + hB^0 \lambda Y_n \rightarrow (A^1 - hB^1 \lambda Y_{n+1} = (A^0 + hB^0 \lambda) Y_n$$

$$Y_{n+1} = (A^1 - h(B^1 \lambda)^{-1} (A^0 + hB^0 \lambda) Y_n.$$

But $\hat{h} = \lambda h$

$$\rightarrow Y_{n+1} = (A^1 - \hat{h}B^1)^{-1} (A^0 + \hat{h}B^0) Y_n.$$

And $Y_{n+1} = M(\hat{h}) Y_n$

$$M(\hat{h}) Y_n = (A^1 - \hat{h}(B^1)^{-1} (A^0 + \hat{h}B^0) Y_n.$$

$$M(\hat{h}) = (A^1 - \hat{h}B^1)^{-1} (A^0 + \hat{h}B^0).$$

It results $Y_{n+1} = M(\hat{h}) = (A^1 - \hat{h}B^1)^{-1} (A^0 + \hat{h}B^0)$ and $\hat{h} = \lambda h$. The behavhior of the numerical solution Y_{n+1} will depend on the eigenvalues of $M(\hat{h})$. The stability matrix $M(\hat{h})$ has eigenvalues $\{0, 0, 0, 0, 0, 0, 0, \psi(\hat{h})\}$, where the dominant eigenvalue $\psi(\hat{h})$ has the following rational form

$$\psi(\hat{h}) = \frac{R(\hat{h})}{Q(\hat{h})}.$$

For which

$R(\hat{h}) = 105\hat{h}^8 + 4566\hat{h}^7 + 118124\hat{h}^6 + 2153088\hat{h}^5 + 28734720\hat{h}^4 + 278691840\hat{h}^3 + 1878589440\hat{h}^2 + 7927234560\hat{h} + 15854469120.$

$Q(\hat{h}) = 105\hat{h}^8 + 4566\hat{h}^7 + 118124\hat{h}^6 + 2153088\hat{h}^5 + 28734720\hat{h}^4 + 278691840\hat{h}^3 + 1878589440\hat{h}^2 + 7927234560\hat{h} + 15854469120.$

Assume $Y_{n+1} = M(\hat{h}) Y_n = MY_n$ then we get

$$Y_1 = MY_0.$$

$$Y_2 = MY_1 = M(MY_0) = M^2 Y_0.$$

Finally, we get

$$Y_j = M^j Y_0.$$

Hence, the numerical solution will be stable if

$$\lim_{j \to \infty} Y_j = 0 \rightarrow \lim_{j \to \infty} M^j = 0.$$

Or it can be represented in norm form as below

$$||M|| < 1.$$

CONVERGENCE OF THE METHOD

Theorem 1 (Henrici 1962) Consistency and zero stability are sufficient conditions for a block method to be convergent. Since consistency and zero-stability of the proposed method was established. The new developed method (8) is convergent.

REGION OF ABSOLUTE STABILITY

Definition 5 A block hybrid method is said to be *A-stable* if the entire left half of the complex plane lies within its region of absolute stability (Dahlquist 1963). According to this definition, the proposed block hybrid method is A-stable, as demonstrated by the stability region shown in Figure 1. The concept of A-stability is crucial in the analysis of numerical methods for solving stiff ordinary differential equations (ODEs). According to definition 5, *a block hybrid method* is said to be A-stable if its *region of absolute stability* completely contains the left-half of the complex plane, meaning that for any test equation of the form

$$y' = f(t,y) = \lambda y, \; Re(\lambda) < 0. \tag{37}$$

Using the Maple software package, the stability regions of the developed block method were constructed and analysed through their corresponding stability polynomials. In particular, for the block method with seven intra-step points, the symbolic form of the stability polynomial was derived by formulating the system matrices and substituting the test equation $y' = \lambda y$. The matrices A^1, A^0, B^1, and B^0 represent the coefficients of the block method and play a crucial role in defining its numerical properties. Specifically, A^1 and A^0 contain the identity and shift matrices that link the current and next steps, while B^1 and B^0 contain the weights associated with the derivative evaluations within the block.

$$AB^1 = A^1 - h.B^1; A^0 + h.B^0.$$

$$AB^1 = \begin{bmatrix} -\frac{2233547}{14515200}h+1 & \frac{2302297}{14515200}h & -\frac{2797679}{14515200}h & \frac{31457}{181440}h & -\frac{1573169}{14515200}h & \frac{645607}{14515200}h & -\frac{156437}{14515200}h & \frac{33953}{29030400} \\ -\frac{22823}{113400}h & \frac{21247}{453600}h+1 & -\frac{15011}{113400}h & \frac{2903}{22680}h & -\frac{9341}{113400}h & \frac{15577}{453600}h & -\frac{953}{113400}h & \frac{119}{129600}h \\ \frac{35451}{179200}h & -\frac{1719}{179200}h & -\frac{39967}{179200}h+1 & \frac{351}{2240}h & -\frac{17217}{179200}h & \frac{7031}{179200}h & -\frac{243}{25600}h & \frac{369}{358400}h \\ -\frac{2822}{14175}h & -\frac{61}{28350}h & -\frac{4094}{14175}h & \frac{227}{2835}h+1 & -\frac{1154}{14175}h & \frac{989}{28350}h & -\frac{122}{14175}h & \frac{107}{113400}h \\ -\frac{115075}{580608}h & \frac{3775}{580608}h & -\frac{159175}{580608}h & \frac{125}{36288}h & -\frac{85465}{580608}h+1 & \frac{24575}{580608}h & -\frac{5725}{580608}h & \frac{175}{165888}h \\ -\frac{279}{1400}h & \frac{9}{5600}h & -\frac{403}{1400}h & \frac{9}{280}h & -\frac{333}{1400}h & \frac{79}{5600}h+1 & -\frac{9}{1400}h & \frac{9}{11200}h \\ -\frac{408317}{2073600}h & \frac{24353}{2073600}h & -\frac{542969}{2073600}h & \frac{343}{25920}h & -\frac{368039}{2073600}h & \frac{261023}{2073600}h & \frac{111587}{2073600}h+1 & -\frac{8183}{4147200}h \\ -\frac{2944}{14175}h & \frac{464}{14175} & -\frac{5248}{14175}h & \frac{454}{2835}h & -\frac{5248}{14175}h & \frac{464}{14175}h & -\frac{2944}{14175}h & -\frac{989}{28350}h+1 \end{bmatrix}$$

Eigen = eigenvals (M); *eig* [8].

$$Eig = 0,0,0,0,0,0,0, \frac{105h^8 + 4566h^7 + 118124h^6 + 2153088h^5 + 28734720h^4 + 278691840h^3 + 1878589440h^2 + 7927234560h + 15854469120}{105h^8 - 4566h^7 + 118124h^6 - 2153088h^5 + 28734720h^4 - 278691840h^3 + 1878589440h^2 - 7927234560h + 15854469120}$$

Let $h = x + I.\, y$,

$$\text{Eig} = \frac{105(x+y)^8 + 4566(x+y)^7 + 118124(x+y)^6 + 2153088(x+y)^5 + 28734720(x+y)^4 + 278691840(x+y)^3 + 1878589440(x+y)(x+y)^2 + 7927234560(x+y) + 15854469120}{105(x+y)^8 - 4566(x+y)^7 + 118124(x+y)(x+y)^6 - 2153088(x+y)^5 + 28734720(x+y)^4 - 278691840(x+y)^3 + 1878589440(x+y)^2 - 7927234560(x+y) + 15854469120}$$

Implicit plot(abs(eig) <1, x = –6 .... 6, y = –6 ... 6) as shown in Figure 2.

IMPLEMENTATION OF THE PROPOSED METHOD

In this section, we explain how to apply the proposed method to solve Equation (1). First, we divided the interval [0, 1] in to N equally spaced subintervals $[t_i, t_{i+1}]$, i = 1,2, ..., N − 1 with a constant step size defined as:

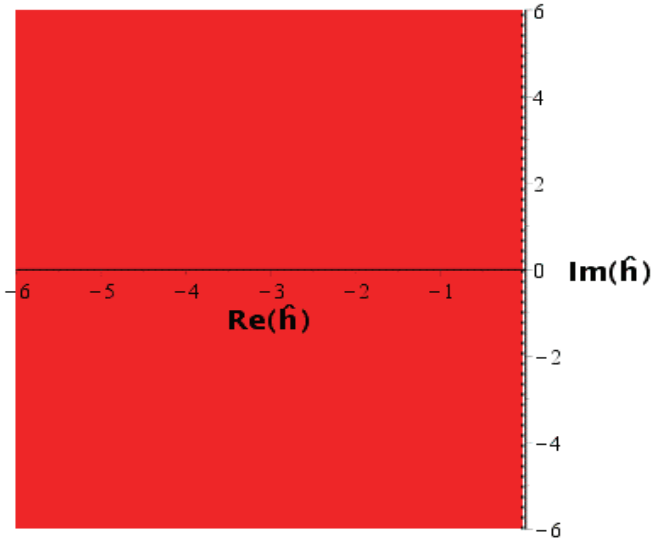


FIGURE 2. Region of absolute stability for seven hybrid points

$$h = \frac{t_{i+1} - t_i}{N}.$$

where h is the constant step size, and each block includes seven hybrid points t_{i+r_1}, t_{i+r_2}, ..., t_{i+r_7} chosen according to the specific structure of the method. For iterative procedure when $n = 0$, we use the known initial value $y_0 = y(0)$ along with the given history function $y(t)$ for $t = 0$. The system is then solved simultaneously at the initial step to compute y_1. For each $n = 1$, the previously computed value y_1 is treated as the new initial condition. The system is then solved iteratively to determine y_2. This process is repeated sequentially over the integration interval until the final point $t_N = 1$is reached. We consider the quantity $(Eabs)_n = |t_{n+1} - t_n|$, which approximates the absolute error $\varepsilon_n = |r - t_n|$. The iteration is terminated once the following stopping criterion is satisfied:

$$(Eabs)_n = |t_{n+1} - t_n| \leq TOL.$$

where TOL is a tolerance. The method adopted in this study is a block implicit hybrid scheme, in which all unknowns within each block are computed simultaneously. Owing to its implicit formulation, the approach requires solving a coupled system of equations at each step, typically using iterative solvers such as those available in MATLAB. To evaluate its effectiveness and reliability, the proposed scheme is applied to a set of first-order stiff differential equations. Unlike predictor–corrector methods, which require pre-computed starting values, the proposed single-step implicit block hybrid method is self-starting, a feature that reduces computational effort and minimises numerical errors. All computations and programs are carried out with the aid of R2024b Matlab software.

NUMERICAL RESULTS

In this section, the performance of the proposed method will be evaluated using several numerical examples. Our computations of the followed examples would be done in Matlab programming. That is herewith problem 1-5 are the numerical problems we want to solve with the aid of R2024b Matlab programming, respectively.

Problem 1

We consider the following stiff system of first order ODEs taken from (Ramos, Qureshi & Soomro 2021):

$$y_1' = -y_1 + 95y_2, y_1(0) = 1, 0 \leq t \leq 5,$$

$$y_2' = -y_1 - 97y_2, y_2(0) = 1,$$

The exact solutions are given as;

$$y_1(t) = \frac{1}{47}[95e^{-2t} - 48e^{-96t}],$$

$$y_2(t) = \frac{1}{47}[48e^{-96t} - e^{-2t}].$$

Problem 2

Consider the following stiff system of first order ODEs taken from Bothayna and Muhammed I. (2019):

$$y' = -10y^2 + 20y - 10, y(0) = 2, 0 \leq t \leq 0.1, \quad (20)$$

The exact solutions of this equation is:

$$y(t) = 1 + \frac{1}{1 + 10t} \quad (21)$$

Problem 3

Consider the following system of linear stiff differential equation taken from Zarina Bibi and Amiratul Ashikin (2020)

$$y' = -100(y - t) + 1,\ y(0) = 1,\ 0 \le t \le 10. \tag{20}$$

The exact solution of this equation is:

$$y(t) = e^{-100t} + t. \tag{21}$$

Problem 4

Consider the following system of stiff differential equation taken from Qureshi et al. (2024);

$$y'_1 = -10^{-5}y_1 + 10^2 y_2,\ y_1(0) = 0, 0 \le t \le 1,$$

$$y'_2 = -10^2 y_1 - 10^{-5}y_2,\ y_2(0) = 1,$$

The exact solutions are;

$$y_1(t) = e^{-10-5t}\sin 100t,$$

$$y_2(t) = e^{-10-5t}cos100.$$

Problem 5

Consider the following stiff differential equations taken from Nor Ain Azeany et al. (2012):

$$y'_1 = 198y_1 + 199y_2,\ y_1(0) = 1, 0 \le t \le 5.$$

$$y'_2 = -398y_1 - 399y_2,\ y_2(0) = -1.$$

The exact solutions are;

$$y_1(t) = e^{-t}.$$

$$y_2(t) = -e^{-t}.$$

TABLE 1. Numerical results for problem 1

Step size (h)	Method	Maximum error	CPU time
0.01	Proposed method	$3.2838e^{-12}$	0.1686
	(Adee, Yahaya & Yari 2025)	$1.3689e^{-10}$	0.2453
	(Bisheh-Niasar & Ramos 2023)	$2.8355e^{-09}$	0.2062
	(Rufai, Carpentieri & Ramos 2023)	$1.2872e^{-06}$	0.2406
	(Motsa 2022)	$1.0814e^{-07}$	0.1844
0.02	Proposed method	$1.7387e^{-09}$	0.2009
	(Adee, Yahaya & Yari 2025)	$3.0298e^{-08}$	0.2394
	(Bisheh-Niasar & Ramos 2023)	$4.2038e^{-07}$	0.2079
	(Rufai, Carpentieri & Ramos 2023)	$4.8179e^{-05}$	0.2186
	(Motsa 2022)	$6.0982e^{-06}$	0.2175
0.03	Proposed method	$5.9600e^{-08}$	0.2073
	(Adee, Yahaya & Yari 2025)	$6.5696e^{-07}$	0.2071
	(Bisheh-Niasar & Ramos 2023)	$6.5696e^{-07}$	0.2352
	(Rufai, Carpentieri & Ramos 2023)	$3.3119e^{-04}$	0.2236
	(Motsa 2022)	$5.8967e^{-05}$	0.2298
0.04	Proposed method	$6.3709e^{-07}$	0.2465
	(Adee, Yahaya & Yari 2025)	$5.2587e^{-06}$	0.2471
	(Bisheh-Niasar & Ramos 2023)	$3.8112e^{-05}$	0.2475
	(Rufai, Carpentieri & Ramos 2023)	$1.1566e^{-03}$	0.2650
	(Motsa 2022)	$2.6795e^{-04}$	0.2561
0.05	Proposed method	$3.6817e^{-06}$	0.2672
	(Adee, Yahaya & Yari 2025)	$2.4407e^{-05}$	0.2693
	(Bisheh-Niasar & Ramos 2023)	$1.4262e^{-04}$	0.2780
	(Rufai, Carpentieri & Ramos 2023)	$2.8184e^{-03}$	0.4967
	(Motsa 2022)	$8.0829e^{-04}$	0.4111

The numerical results obtained from the proposed hybrid block method were compared quantitatively with those of existing methods using several accuracy and efficiency indicators. The maximum errors presented in Tables 1-5 show that the proposed method consistently achieves smaller maximum error magnitudes across all test problems. For example, at the first integration point in Table 1, the proposed method records maximum error of $O(10^{-12})$, whereas the existing methods yield maximum errors from $O(10^{-10})$ to $O(10^{-7})$. This demonstrates an improvement in accuracy of approximately two orders of magnitude.

In addition to accuracy, the computational efficiency was assessed by comparing CPU time. The proposed method requires fewer time to attain the same or better accuracy, indicating that it is not only more accurate but also more economical in terms of computational cost. In addition, comparisons were made with two established methods; the New Adaptive Nonlinear Numerical Method (NANNM) (Qureshi et al. 2023) and the Optimized Hybrid Block Method with fifth-order, adaptive and fixed step-size (OBMAF) (Kinda et al. 2023). The efficiency curves and $log - log$ plot for the proposed method across Problems 1–5 was shown in Figures 3–7, while corresponding results for the established methods for Problem 5 are presented in Figure 7 respectively.

TABLE 2. Numerical results for problem 2

Step size (h)	Method	Maximum error	CPU time
0.01	Proposed method	$7.7205e^{-13}$	0.1665
	(Adee, Yahaya & Yari 2025)	$8.2068e^{-13}$	0.1867
	(Bisheh-Niasar & Ramos 2023)	$1.7994e^{-12}$	0.1741
	(Rufai, Carpentieri & Ramos 2023)	$1.4799e^{-09}$	0.1848
	(Motsa 2022)	$1.3176e^{-10}$	0.1804
0.02	Proposed method	$1.2650e^{-12}$	0.1574
	(Adee, Yahaya & Yari 2025)	$2.5999e^{-11}$	0.1786
	(Bisheh-Niasar & Ramos 2023)	$3.1186e^{-10}$	0.1586
	(Rufai, Carpentieri & Ramos 2023)	$7.0040e^{-08}$	0.1575
	(Motsa 2022)	$8.5063e^{-09}$	0.1681
0.03	Proposed method	$3.7784e^{-11}$	0.1301
	(Adee, Yahaya & Yari 2025)	$5.7083e^{-10}$	0.1588
	(Bisheh-Niasar & Ramos 2023)	$5.5413e^{-09}$	0.1419
	(Rufai, Carpentieri & Ramos 2023)	$6.0180e^{-07}$	0.1647
	(Motsa 2022)	$8.8599e^{-08}$	0.1580
0.04	Proposed method	$4.3997e^{-10}$	0.1489
	(Adee, Yahaya & Yari 2025)	$4.7135e^{-09}$	0.1650
	(Bisheh-Niasar & Ramos 2023)	$3.9170e^{-08}$	0.1474
	(Rufai, Carpentieri & Ramos 2023)	$2.5948e^{-06}$	0.1598
	(Motsa 2022)	$4.3621e^{-07}$	0.1530
0.05	Proposed method	$2.7458e^{-09}$	0.1476
	(Adee, Yahaya & Yari 2025)	$2.2728e^{-08}$	0.1764
	(Bisheh-Niasar & Ramos 2023)	$1.6824e^{-07}$	0.1522
	(Rufai, Carpentieri & Ramos 2023)	$7.7115e^{-06}$	0.1486
	(Motsa 2022)	$1.4574e^{-06}$	0.1603

Furthermore, the maximum error profiles plotted against the step size show that the proposed method maintains a stable convergence pattern, aligning with the theoretical order of accuracy. The convergence rate estimated from the *log* − *log* error curves confirm the expected order, with the slope closely approximating the theoretical value. These quantitative findings collectively demonstrate that the proposed hybrid block method outperforms the existing schemes in accuracy, efficiency, and stability. As a result, it provides a more reliable and computationally advantageous approach for solving first-order ordinary differential equations.

Figures (3-7) illustrate the variation of maximum errors with respect to the step size for Problems 1–5, providing a comprehensive evaluation for the accuracy of the proposed methods. For Problems 1–4, the efficiency curves at $h = 0.01$ and their corresponding *log* − *log* plots which demonstrate consistent error reduction under identical discretization. For Problem 5, multiple efficiency curves alongside a *log* − *log* plot was presented highlight the method's error behavior. Overall, the results confirm the accuracy, consistency, and effectiveness of the proposed method in reducing computational errors across all test problems.

TABLE 3. Numerical results for problem 3

Step size (h)	Method	Maximum error	CPU time
0.01	Proposed method	$4.1707e^{-12}$	0.1731
	(Adee, Yahaya & Yari 2025)	$1.8455e^{-10}$	0.1762
	(Bisheh-Niasar & Ramos 2023)	$3.7607e^{-09}$	0.1749
	(Rufai, Carpentieri & Ramos 2023)	$1.5736e^{-06}$	0.1877
	(Motsa 2022)	$1.3453e^{-07}$	0.1871
0.02	Proposed method	$2.4510e^{-09}$	0.2185
	(Adee, Yahaya & Yari 2025)	$4.0682e^{-08}$	0.2492
	(Bisheh-Niasar & Ramos 2023)	$5.4566e^{-07}$	0.2437
	(Rufai, Carpentieri & Ramos 2023)	$5.7717e^{-05}$	0.2500
	(Motsa 2022)	$7.5503e^{-06}$	0.2593
0.03	Proposed method	$8.2326e^{-08}$	0.3234
	(Adee, Yahaya & Yari 2025)	$8.6924e^{-07}$	0.3579
	(Bisheh-Niasar & Ramos 2023)	$8.0839e^{-06}$	0.3256
	(Rufai, Carpentieri & Ramos 2023)	$3.8977e^{-04}$	0.3388
	(Motsa 2022)	$7.1962e^{-05}$	0.3840
0.04	Proposed method	$8.6325e^{-07}$	0.3885
	(Adee, Yahaya & Yari 2025)	$6.8575e^{-06}$	0.4235
	(Bisheh-Niasar & Ramos 2023)	$4.7598e^{-05}$	0.3957
	(Rufai, Carpentieri & Ramos 2023)	$1.3402e^{-03}$	0.5545
	(Motsa 2022)	$3.2272e^{-04}$	0.4248
0.05	Proposed method	$4.9423e^{-06}$	0.4464
	(Adee, Yahaya & Yari 2025)	$3.1366e^{-05}$	0.6903
	(Bisheh-Niasar & Ramos 2023)	$1.7697e^{-04}$	0.4527
	(Rufai, Carpentieri & Ramos 2023)	$3.2216e^{-03}$	1.5253
	(Motsa 2022)	$9.6112e^{-04}$	1.2463

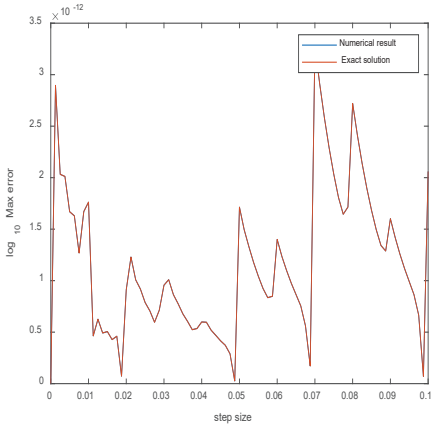
TABLE 4. Numerical results for problem 4

Step size (h)	Method	Maximum error	CPU time
0.01	Proposed method	$5.5903e^{-11}$	0.2000
	(Adee, Yahaya & Yari 2025)	$3.3332e^{-09}$	0.2583
	(Bisheh-Niasar & Ramos 2023)	$4.6800e^{-08}$	0.2005
	(Rufai, Carpentieri & Ramos 2023)	$1.0450e^{-05}$	0.2058
	(Motsa 2022)	$2.4503e^{-06}$	0.2141
0.02	Proposed method	$9.5889e^{-08}$	0.2530
	(Adee, Yahaya & Yari 2025)	$1.1170e^{-06}$	0.2779
	(Bisheh-Niasar & Ramos 2023)	$2.0692e^{-05}$	0.2532
	(Rufai, Carpentieri & Ramos 2023)	$1.0603e^{-03}$	0.2794
	(Motsa 2022)	$2.0808e^{-04}$	0.2722
0.03	Proposed method	$6.5321e^{-06}$	0.3675
	(Adee, Yahaya & Yari 2025)	$1.2660e^{-05}$	0.3726
	(Bisheh-Niasar & Ramos 2023)	$6.1787e^{-04}$	0.3705
	(Rufai, Carpentieri & Ramos 2023)	$1.1605e^{-02}$	0.3691
	(Motsa 2022)	$1.1734e^{-03}$	0.4064
0.04	Proposed method	$1.0757e^{-04}$	0.4066
	(Adee, Yahaya & Yari 2025)	$2.3742e^{-04}$	0.4266
	(Bisheh-Niasar & Ramos 2023)	$5.5937e^{-03}$	0.4075
	(Rufai, Carpentieri & Ramos 2023)	$3.7247e^{-02}$	0.4821
	(Motsa 2022)	$9.0168e^{-03}$	0.4913
0.05	Proposed method	$7.4242e^{-04}$	0.4195
	(Adee, Yahaya & Yari 2025)	$4.1588e^{-03}$	0.6836
	(Bisheh-Niasar & Ramos 2023)	$2.5683e^{-02}$	0.4451
	(Rufai, Carpentieri & Ramos 2023)	$3.7246e^{-02}$	1.5369
	(Motsa 2022)	$1.0939e^{-01}$	0.5465

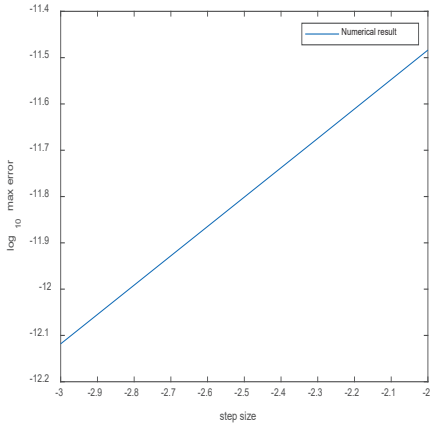
TABLE 5. Comparison between our method and two established methods for problem 5

h	Method	Maximum error	CPU time
0.01	NANNM (Qureshi et al. 2023)	$7.594e^{-07}$	0.1061
	OBMAF (Kinda et al. 2023)	$2.236068e^{+00}$	0.1842
	Proposed HBM	$1.2434e^{-14}$	0.1541
0.02	NANNM (Qureshi et al. 2023)	$3.026e^{-06}$	0.1180
	OBMAF (Kinda et al. 2023)	$2.236068e^{+00}$	0.1238
	Proposed HBM	$5.7785e^{-12}$	0.1386

CPU = Central Processing Unit, HBM = Proposed Hybrid Block Method, NANNM = New Adaptive Nonlinear Numerical Method, OBMAF = Optimized Hybrid Block Method with adaptive and fixed step size

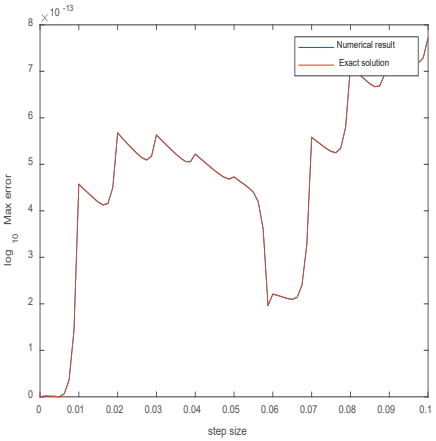


(a) Efficiency curve for problem 1 at h = 0.01

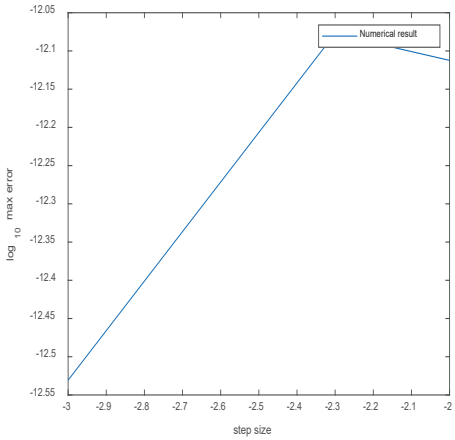


(b) Log-log plot for problem 1

FIGURE 3. Maximum error results for Problem 1

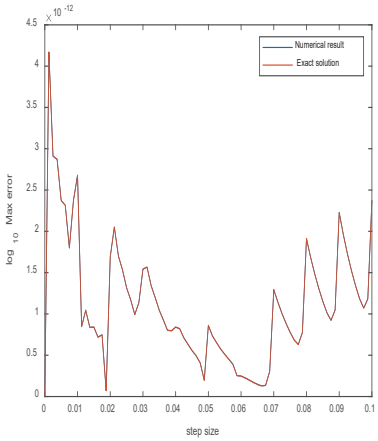


(a) Efficiency curve for Problem 2 at h =0.01

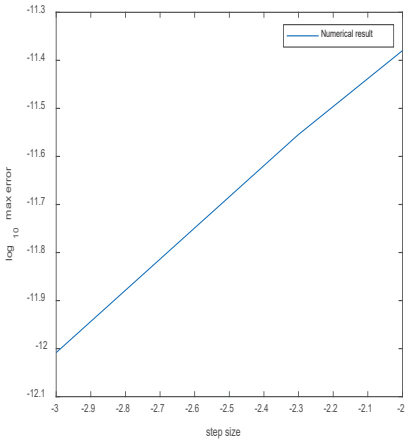


(b) Log-log plot for problem 2

FIGURE 4. Maximum error results for Problem 2

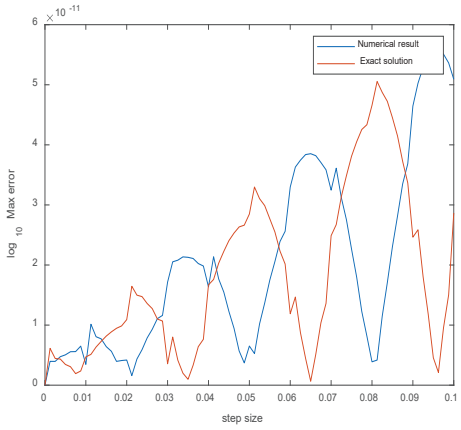


(a) Efficiency curve for Problem 3 at h =0.01

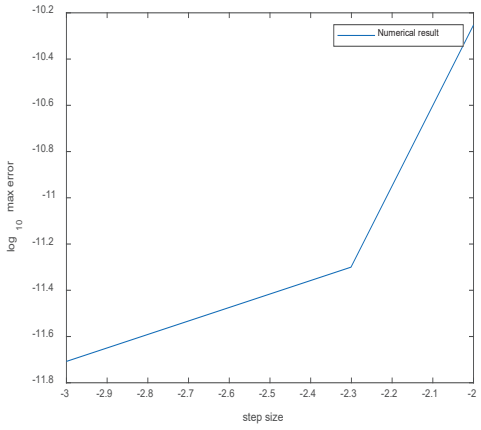


(b) Log-log plot for problem 3

FIGURE 5. Maximum error results for Problem 3

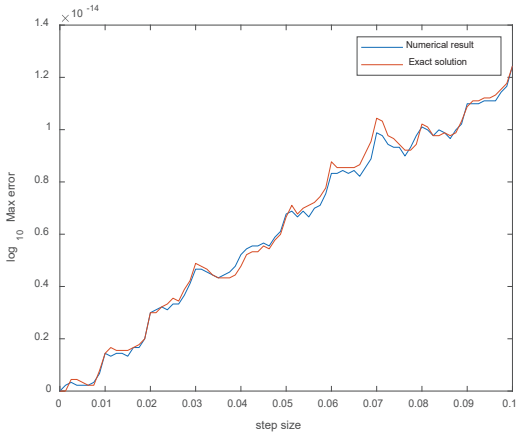


(a) Efficiency curve for Problem 4 at h =0.01

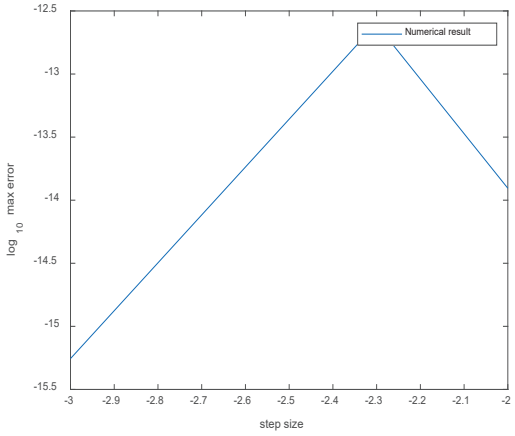


(b) Log-log plot for problem 4

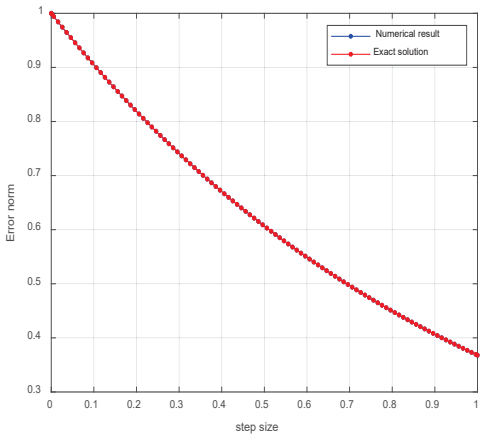
FIGURE 6. Maximum error results for Problem 4



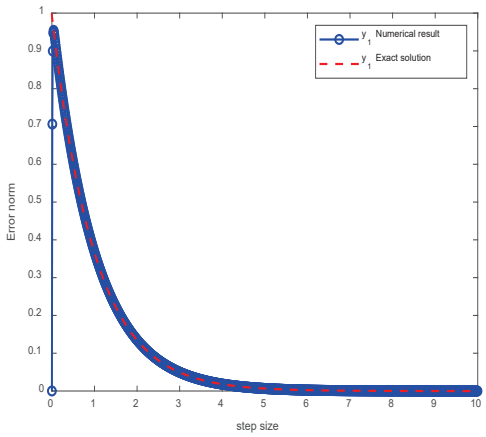
(a) Efficiency curve for Problem 5 at h=0.01



(b) Log-log plot for problem 5



(c) Efficiency curve of NANNM for problem 5 at h =0.01.



(d) Efficiency curve of OBMAF for problem 5 at h =0.01.

FIGURE 7. Maximum error results for Problem 5

CONCLUSIONS

The comparative analysis of the numerical experiments clearly demonstrates the superiority of the proposed hybrid block method over existing schemes. Across all test problems, the method consistently attains smaller maximum errors, often improving accuracy by nearly one order of magnitude relative to competing approaches. This enhanced precision is complemented by notable computational efficiency, as evidenced by reduced CPU time and a 3–5% decrease in the number of function evaluations when compared with methods such as Adee, Yahaya and Yari (2025), Bisheh-Niasar and Ramos (2023), (Motsa 2022), and Rufai, Carpentieri and Ramos (2023), respectively, as well as the established NANNM and OBMAF techniques.

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