Chapter 1

Fundamental Concepts

1.1 Introduction

The finite element method (FEM) has become a powerful tool for the numerical solution of a wide range of engineering problems. Application ranges from deformation and stress analysis of automotive, aircraft, building and bridge structures to field analysis of heat flux, fluid flow, magnetic flux, seepage and other flow problems. Research about its application is still open and carried out which also covers non engineering fields such as economic analysis, optimization and so on.

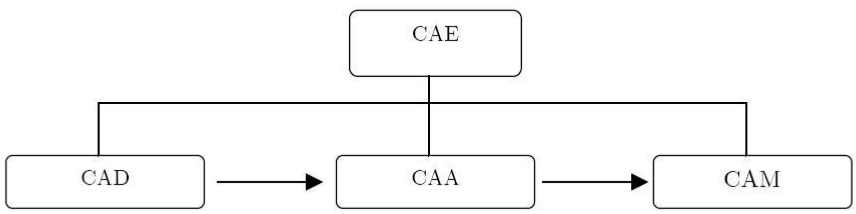

Figure 1:

Relationship between CAD, CAE and CAM is shown in Figure 1

1.2 Historical Background

In 1941, Hrenikoff presented a solution of elasticity problems using the 'frame work method' to analyse aircraft structure. In 1960, the term finite element was introduced by Clough. The first book on finite element by Zienkiewicz and Cheung was published in 1967. By late 1960s and early 1970s, finite element analysis was applied to nonlinear problems and large deformations. A book on nonlinear continua appeared in 1972 by oden. Today, the development in mainframe computers and avaibility of powerful microcomputers has brought this method within reach of students and engineers.

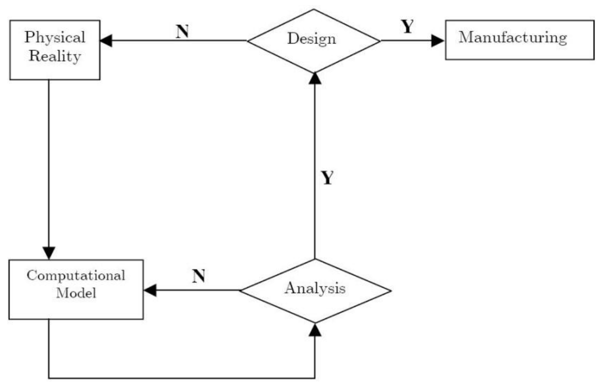

Figure 2:

Computer plays an important role in FEM. Nowadays, many FEM commercial packages can be found such as Pro-Engineer, COSMOS/M, ALGOR, LUSAS and others. Its provide easy use of input and output facilities. However, this package is not provided with source code and solution method. Because of this reason, the learning process as engineer should cover the theory development so that the total solution and result interpretation can be done correctly.

1.3 Stresses and Equilibrium

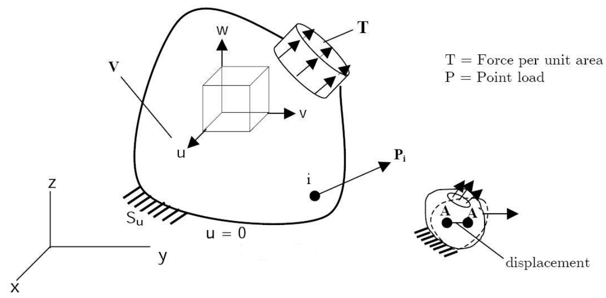

Figure 3:

By considering a three dimensional solid body as shown in Figure 1.3.

Under the force, the body deforms. The deformation of a point $x\left(= [x,\, y,\, z]^{\mathrm{T}}\right)$**, is given by the three components of its displacement:**

$$\mathrm{u} = [u,\, v,\, w]^{\mathrm{T}}$$

The distributed force per unit volume, for example, the weight per unit volume, is the vector f which is given by:

$$\mathrm{f} = [f_x,\, f_y,\, f_z]^{\mathrm{T}}$$

The surface traction $\mathbf{T}$**, for example, the contact force or action of pressure is given by:**

$$\mathrm{T} = [T_x,\, T_y,\, T_z]^{\mathrm{T}}$$

A load , P acting at a point i is represented by its three components :

From Figure 1.4 above, stresses acting on point x is given by:

$$\sigma = [\sigma_x,\, \sigma_y,\, \sigma_z,\, \tau_{yz},\, \tau_{xz},\, \tau_{xy}]^{\mathrm{T}}$$

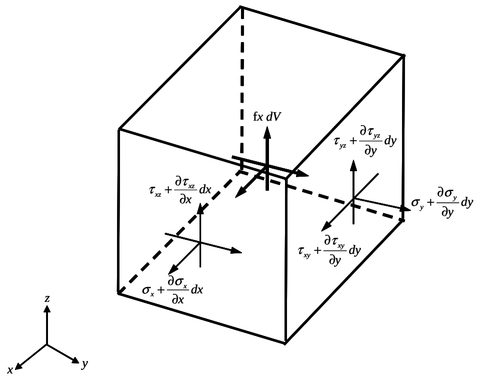

Figure 4:

where σ_x, σ_y, σ_z**are normal stresses and** τ_{yz}, τ_{xz}, τ_{xy}**are shear stresses. Let us consider equilibrium of the elemental volume shown in Figure 1.4. Writing** $\sum F_x = 0$, $\sum F_y = 0$**and** $\sum F_z = 0$**and recognizing** $dV = dxdydz$ **we get the equilibrium equation:**

$$\frac{\partial \sigma_x}{\partial_x} + \frac{\partial \tau_{xy}}{\partial_y} + \frac{\partial \tau_{xy}}{\partial_z} + f_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial_x} + \frac{\partial \sigma_y}{\partial_y} + \frac{\partial \tau_{yz}}{\partial_z} + f_y = 0$$

$$\frac{\partial \tau_{xz}}{\partial_x} + \frac{\partial \tau_{yz}}{\partial_y} + \frac{\partial \sigma_z}{\partial_z} + f_z = 0$$

1.4 Boundary Conditions

Referring to Figure 1.4, we find that there are displacement boundary conditions and surface-loading conditions. If there are no displacement on part of the boundary denoted by S_u **we have:**

$$u = 0 \text{on } S_u$$

We can also consider boundary conditions such as $u = \mathrm{a}$ **where** a**is a given displacement.**

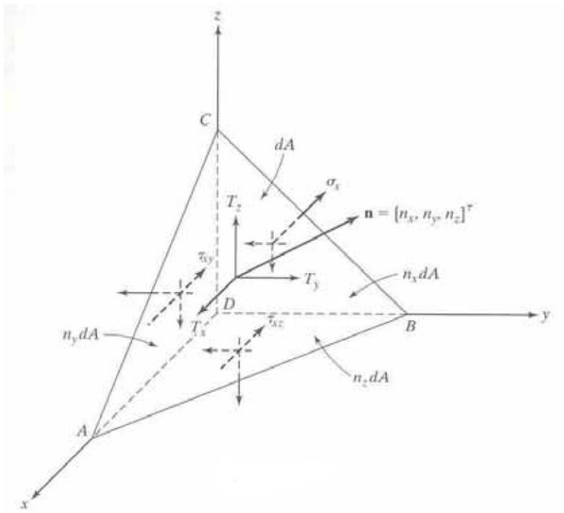

Figure 5:

Consider the tetrahedron elemental as shown in Figure 1.5. Area is denoted by dA. If $n = [nx, ny, nz]^{\mathrm{T}}$is the unit normal to dA, then area $BDC = n_x dA$,area $ADC = n_y dA$ and area $ADB = n_z dA$. Consideration of equilibrium along the three axes directions gives $\sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z = T_x$

$\tau_{xy} n_x + \sigma_y n_y + \tau_{yz} n_z = T_y$

$\tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z = T_z$

These conditions must be satisfied on the boundary, S_T, where the tractions are applied.

1.5 Strain-Displacement Relations

We represent the strains in a vector form

$$\epsilon = [\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{yz}, \gamma_{xz}, \gamma_{xy}]^{\mathrm{T}}$$

where $\epsilon_x, \epsilon_y, \epsilon_z$ and are normal strains and are$\gamma_{yz}, \gamma_{xz}, \gamma_{xy}$ the shear strains. We can also write these strains by

$$\epsilon = \left[\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}, \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]^{\mathrm{T}}$$

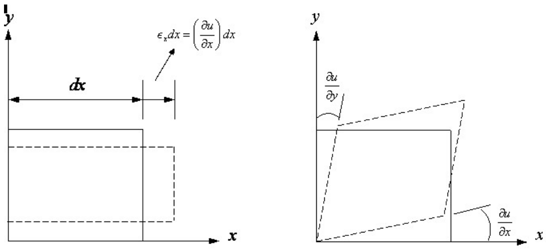

Figure 6:

These strain relations hold for small deformations.

1.6 Stress-Strain Relations

For linear elastic materials, the stress-strain relations come from the generalized Hooke's law.

$$\epsilon_x = \frac{\sigma_x}{E} - v\frac{\sigma_y}{E} - v\frac{\sigma_z}{E}$$

$$\epsilon_y = \frac{\sigma_x}{E} - v\frac{\sigma_y}{E} - v\frac{\sigma_z}{E}$$

$$\epsilon_z = \frac{\sigma_x}{E} - v\frac{\sigma_y}{E} - v\frac{\sigma_z}{E}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G}$$

$$\gamma_{xz} = \frac{\tau_{xz}}{G}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

where G is the shear modulus (or modulus of rigidity) is given by:

$$G = \frac{E}{2(1+v)}$$

Relations that can be introduced is

$$\sigma = D\epsilon$$

where D is the symmetric (6 x 6) material matrix given by:

$$D = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v & 0 & 0 & 0 \\ v & 1-v & v & 0 & 0 & 0 \\ v & v & 1-v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5-v & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5-v & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5-v \end{bmatrix}$$

For one dimensional cases, stress-strain relations are simply:

$$\sigma = E\epsilon$$

While for two dimensional cases, the problems are modeled as plane stress and plane strain.

1.6.1 Plane Stress.

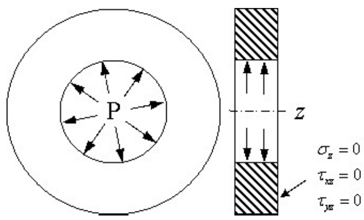

Figure 7:

A thin planar body subjected to in-plane loading on its edge surface is said to be in plane stress. A ring press fitted on a shaft as shown in Figure 1.7 as example. Here, stresses σ_z, τ_{xz} and τ_{yz} are set as zero.

The material matrix becomes:

$$D = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \frac{1}{2}-v \end{bmatrix}$$

1.7 Temperature Effects

For isotropic material, the temperature rise ΔT results in a uniform strain. It depend on the coefficient of linear expansion α of the material. This coefficient represent the change in length per unit temperature rise. The temperature strain is represented as an initial strain:

$$\epsilon 0 = [\alpha\Delta T,\, \alpha\Delta T,\, \alpha\Delta T,\, 0,\, 0,\, 0]^{\mathrm{T}}$$

The stess-strain relations then become:

$$\sigma = D\,(\epsilon - \epsilon_0)$$

In plane stress, we have:

$$\epsilon 0 = \left[[\alpha\Delta T,\, \alpha\Delta T,\, 0]^{\mathrm{T}}\right]$$

In plane strain we have:

$$\epsilon 0 = (1+v)\,[\alpha\Delta T,\, \alpha\Delta T,\, 0]^{\mathrm{T}}$$

Solution of this set of equations is generally referred to as an exact solution. Such exact solution are available for simple geometries and loading conditions. For problems of complex geometries and general boundary and loading conditions, obtaining such solutions is an almost impossible task. Because of that, approximate solution method usually employ potential energy methods or variational methods to solve the problem.

1.8 Potential Energy

The total potential energy of an elastic body, is defined as the sum of total strain energy and the work potential which is given as:

$$\Pi = \frac{1}{2}\int_V \sigma^{\mathrm{T}}\epsilon\, dV - \int_V \mathrm{u}^{\mathrm{T}}\mathrm{f}\, dv - \int_S \mathrm{u}^{\mathrm{T}}\mathrm{T}\, dS - \sum_i u_i^{\mathrm{T}}\mathrm{P}_i$$

1.9 Variational Method

Variational Method or the weak form is derived from the Galerkin's method which give:

$$\int_V \sigma^{\mathrm{T}} \epsilon\,(\phi)\; dV - \int_V \phi^{\mathrm{T}} \mathrm{f}\, dV - \int_S \phi^{\mathrm{T}} \mathrm{T}\, dS - \sum_i \phi^{\mathrm{T}} \mathrm{P} = 0$$

where ϕ is an arbitrary displacement consistent with the specified boundary conditions of u. **This equation is also known as Principle of Virtual Work.**

1.10 Von Mises Stress

Von Mises Stress is used as a criterion in determining the onset failure in ductile materials. This criteria states that the Von Mises Stress should be less than the yield stress of the material.

$$\sigma_{VM} \leq \sigma_Y$$

Where the von misses is given by

$$\sigma_{VM} = \sqrt{I_1^2 - 3I_2}$$

With I_1 and I_2 are the first two invariants of the stress tensor.

$$I_1 = \sigma_x + \sigma_y + \sigma_z \text{ or } I_1 = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{yz}^2 - \tau_{xz}^2 - \tau_{xy}^2 \text{ or } I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$

It can be expressed in an easier way by:

$$\sigma_{VM} = \tfrac{1}{2}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

1.11 Matrix Algebra and Gaussian Elimination

1.11.1 Matrix Algebra

The study of matrices is largely motivated from the need to solve systems of simultaneous equations of the form:

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2$$

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$$a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n = b_n$$

where $x_1, x_2, \cdots, x_n$**are the unknowns. Equation above can be conveniently expressed in matrix form as**

$$\mathrm{Ax} = \mathbf{b}$$

The analysis of engineering problems by the finite element method involves a sequence of matrix operations. Gaussian elimination will be applied to solve these simultaneous equation. Before that, explanations about matrix is expressed below.

1.11.2 Row and Column Vectors

A matrix of dimension $(1 \times n)$ **is called a row vector. For example:**

$$\mathrm{d} = \left[\begin{array}{ccc} 1 & -1 & 3 \end{array} \right]$$

A matrix of dimension $(m \times 1)$ **is called a row vector. For example:**

$$K = \left\{ \begin{array}{c} 2 \\ 4 \\ -1 \end{array} \right\}$$

1.11.3 Addition and Subtraction

Matrix A and B can be added or subtracted if both have the same dimension $(m \times n)$. The sum $\mathbf{C} = \mathbf{A} + \mathbf{B}$ is defined as

$$c_{ij} = a_{ij} + b_{ij}$$

Subtraction is similarly defined.

1.11.4 Multiplication by a Scalar

The multiplication of a matrix A by a scalar c is defined as

$$c\mathrm{A} = [ca_{ij}]$$

1.11.5 Matrix Multiplication

The product of an $(m \times n)$ matrix A and an $(n \times p)$ matrix B results in an $(m \times p)$ matrix C. It should be noted that $\mathrm{AB} \neq \mathrm{BA}$.

1.11.6 Transposition

Transpose of a matrix $\mathrm{A} = a_{ij}$ will give . In general if $\mathrm{A}^{\mathrm{T}} = a_{ij}$ is of dimension $(m \times n)$, then AT is of dimension $(n \times m)$. For example:

$$\mathrm{A} = \begin{bmatrix} 1 & -5 \\ 0 & 6 \\ -2 & 3 \\ 4 & 2 \end{bmatrix} \quad \text{then} \quad \mathrm{A}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & -2 & 4 \\ -5 & 6 & 3 & 2 \end{bmatrix}$$

1.11.7 Diagonal Matrix

A diagonal matrix is a square matrix with nonzero elements only along the principal diagonal. For example:

$$\mathrm{A} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1.11.8 Identity Matrix

The identity (or unit) matrix is a diagonal matrix with 1's along the principal diagonal. For example:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1.11.9 Symmetric Matrix

A symmetric matrix is a square matrix whose elements satisfy

$$a_{ij} = a_{ji} \text{ or } \mathrm{A} = \mathrm{A}^{\mathrm{T}}$$

1.11.10 Determinant of a Matrix

The determinant of a square matrix A denoted as $\det \mathbf{A}$**. For example we have a** (3×3) **matrix, then**

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} =$$
$$a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) - a_{12}\left(a_{21}a_{33} - a_{23}a_{31}\right) + a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right)$$

1.11.11 Matrix Inversion

Consider a square matrix A. If det A $\neq 0$**, then A has an inverse, denoted by** $\mathbf{A}^{-1}$**. The inverse satisfies the relations**

$$\mathrm{A}^{-1}\mathrm{A} = \mathrm{A}\mathrm{A}^{-1} = \mathrm{I}$$

If $\det \mathrm{A} \neq 0$**, then we say that** A **is nonsingular. If** $\det \mathrm{A} = 0$**, then we say that** A **is singular, for which the inverse is not defined.**

1.12 Gaussian Elimination

Gaussian elimination is a method of solving simultaneous equations by successively eliminating unknowns. Consider the simultaneous equations below:

$$4x_1 + 2x_2 - 2x_3 - 8x_4 = 4$$

$$x_1 + 2x_2 + x_3 = 2$$

$$0.5x_1 - x_2 + 4x_3 + 4x_4 = 10$$

$$-4x_1 - 2x_2 - x_4 = 0$$

To find the value of x_1, x_2, $x_3$and x_4, Gaussian Elimination process can be explained with

$$\begin{bmatrix} 4 & 2 & -2 & -8 & 4 \\ 1 & 2 & 1 & 0 & 2 \\ 0.5 & -1 & 4 & 4 & 10 \\ -4 & -2 & 0 & -1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0.5 & 0.5 & -2 & 1 \\ 0 & 1.5 & 1.5 & 2 & 1 \\ 0 & -1.25 & 4.25 & 5 & 9.5 \\ 0 & 6 & -2 & -7 & 4 \end{bmatrix} \longrightarrow$$
$$\begin{bmatrix} 1 & 0.5 & -0.5 & -2 & 1 \\ 0 & 1 & 1 & 1.333 & 0.6667 \\ 0 & 0 & 5.5 & 0.6667 & 10.333 \\ 0 & 0 & -2 & -7 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0.5 & -0.5 & -2 & 1 \\ 0 & 1 & 1 & 1.333 & 0.6667 \\ 0 & 0 & 1 & 1.2121 & 1.8788 \\ 0 & 0 & 0 & -4.5758 & 7.7576 \end{bmatrix}$$

The solutions for the system are

$$x_1 = 0.0794,\ x_2 = -1.0066,\ x_3 = 3.9338 \text{ and } x_1 = -1.6954$$

The problem above can be solve easier and faster by using Gauss Program as shown below:

EQUATION SOLVING USING GAUSS ELIMINATION

Number of Equations

4

Matrix A() in Ax = B

$$\begin{array}{ccccc} 4 & 2 & -2 & -8 & 4 \\ 1 & 2 & 1 & 0 & 2 \\ 0.5 & -1 & 4 & 4 & 10 \\ -4 & -2 & 0 & -1 & 0 \end{array}$$

Right hand side B() in Ax = B

4	2	10	0

Results from Program Gauss

EQUATION SOLVING USING GAUSS ELIMINATION

Solution

1. 0.829493

2. -1.06912

3. 3.308756

4. -1.17972